Coupled Magneto-Mechanical Analysis of Inverter-Driven Electrical Machines Taking Account of Magnetostriction

March 2015

Hassan Ebrahimi

Department of Science and Advanced Technology
Graduate School of Science and Engineering
Saga University
Acknowledgements

My deepest gratitude goes first and foremost to Professor Kazuhiro Muramatsu, my supervisor, for his constant encouragement and guidance. He has walked me through all the stages of doing research the writing of this thesis. Without his consistent and illuminating instruction, this thesis could not have reached its present form.

Secondly, I would like to express my heartfelt gratitude to Professor Eiji Takahashi, Professor Satoru Goto, and Doctor Hiroshi Dozono for their valuable comments and suggestions as the members of this dissertation committee.

Third, I am greatly indebted to Doctor Yanhui Gao, for the helpful discussions and constant supports during my tenure. I would also like to express my thanks to my colleagues; Doctor Shunya Odawara and Doctor Lin Chen, for their invaluable helps.

Special thanks go to Mr. Akihisa Kameari for his helpful scientific comments and advices in moments of facing mind-boggling challenges.

I am also indebted to Doctor Takashi Okitsu, for the magnetic and mechanical designs of IPM motor used in this study, and for his helpful comments on the matter.

Finally I would like to acknowledge the financial support of Science Solutions International Laboratory, Inc. (SSIL) during my study.
To my wife Kartika,

and my daughters Sheeva and Melika.
# TABLE OF CONTENTS

Acknowledgements ........................................................................................................ 2

Chapter 1 Introduction ................................................................................................. 7

1.1 Research Background .............................................................................................. 12
  1.1.1 Considering Mechanical Stress in the Analysis of Electromagnetic Devices .......... 13
  1.1.2 Vibration Analysis of Permanent Magnet (PM) Motor Taking Account of Magnetostriction ................................................................. 15

1.2 Research Purposes .................................................................................................... 18
  1.2.1 Coupled Magneto-Mechanical Analysis .............................................................. 18
  1.2.2 Vibration Analysis of Electric Motors ................................................................. 19

Chapter 2 Magnetic Field Analysis ............................................................................... 23

2.1 Introduction .............................................................................................................. 23

2.2 Maxwell Equations .................................................................................................. 25

2.3 $A - \phi$ Magneto-Dynamic Formulation of Maxwell's Equations ............................ 26

2.4 Field-Circuit Equations ........................................................................................... 28

2.5 Finite Element Discretization of Differential Equations ....................................... 31
  2.5.1 Weighted Residual Galerkin Method ................................................................. 33
  2.5.2 Three-Dimensional Discretization Using Hexahedral Edge Elements ................. 42
  2.5.3 Two-Dimensional Discretization Using Triangular Elements ............................ 48

2.6 Iterative Sparse Matrix Solvers for Finite Element Equations .............................. 54
  2.6.1 Conjugate Gradient (CG) Method ..................................................................... 54
  2.6.2 Incomplete Cholesky Conjugate Gradient (ICCG) Method ............................... 57
  2.6.3 Complex Orthogonal ICCG (COICCG) Method .............................................. 60

2.7 Magnetic Nodal Force Calculation ......................................................................... 61

2.8 Magnetostriction ..................................................................................................... 62

2.9 Magnetostriction Strain and Stress Tensors ............................................................ 64

2.10 Magnetostriction Anisotropy .................................................................................. 65
2.11 Magnetostriction Dependence on Mechanical Stress ........................................ 66
2.12 Magnetostriction Nodal Force Calculation ...................................................... 67
2.13 Core Loss Calculation .......................................................................................... 69

Chapter 3 Static Structural Analysis ................................................................. 72
3.1 Introduction ........................................................................................................ 72
3.2 Equation of Static Equilibrium ............................................................................. 74
3.3 Finite Element Discretization of Equation of Static Equilibrium ................. 76
3.4 Calculation of Mechanical Stress in Magnetostrictive Media ................... 82
3.5 Calculation of Shrink-Fit and Press-Fit Stress Using Thermal Force.......... 83

Chapter 4 Coupled Static Magneto-Mechanical Analysis ................................. 86
4.1 Introduction ........................................................................................................ 86
4.2 Coupled Analysis under Longitudinal Stress ................................................. 88
  4.2.1 Numerical Results ............................................................................................. 92
4.3 Coupled Analysis under Multi-Axial Stress .................................................... 96
  4.3.1 Numerical Results ........................................................................................... 102
    4.3.1.1 Prediction of B-H Curves under Uni-Axial Stress ....................................... 102
    4.3.1.2 Effect of Stress on Distribution Of Magnetic Flux Density ...................... 103
4.4 Summary ............................................................................................................ 105

Chapter 5 Dynamic Structural Analysis ......................................................... 106
5.1 Introduction ........................................................................................................ 106
5.2 Equation of Equilibrium Including Friction and Inertia Terms ................. 107
5.3 Finite Element Discretization of Dynamic Equilibrium Equation ............ 107
5.4 Solution Methods of Motion Equation ............................................................. 109
  5.4.1 Time Domain Methods .................................................................................... 110
    5.4.1.1 Backward Euler Method ......................................................................... 111
    5.4.1.2 Newmark Method ................................................................................... 113
  5.4.2 Frequency Domain Method .......................................................................... 114
Chapter 1 Introduction

Our civilization is driven by energy. Various forms of energy are found in the nature and they can be converted from one form to another. One of the highly demanded forms of energy is the kinetic mechanical energy which serves us in transportation, construction, manufacturing, etc. Although humans have been utilizing heat energy for tens of millennia, by burning wood for example, ancient civilizations were not very skilled in converting it into mechanical form. They were mainly driven by muscles of humans and domestic animals or direct use of wind power and hydro energy in certain areas of activity such as milling and navigation.

The situation remained unchanged till late 18th century when British inventor James Watt patented his 10-horsepower steam engine [2] which ignited the first industrial revolution in Great Britain. The steam engine was able to convert the heat into constant motion, useful for driving manufacturing machinery, trains, ships, etc.

Decades after our civilization enjoyed the comfort brought by Watt’s revolutionary invention, electricity just became visible for commercial use in 1831 [2] with the invention of the first dynamo, a crude electric power generator, by Michael Faraday. Faraday’s dynamo could then be run by steam engines or water turbines to generate continuous electricity. Soon after, light bulbs were commercialized to illuminate our civilization, but the use of electricity did not remain limited to illumination. Due to close similarity between electric generator and electric motor, the invention of electric motors was inevitable. Generators convert motion into electricity which drives motors, and motors convert the electricity back motion. This may seem
unremarkable at first, but the advantage comes from the easy transmission of electricity and convenient application of motors where direct application of steam engine or water turbine was impossible or costly. Electric motors of various types were invented and utilized in the industry and gradually replaced steam engines within decades.

Combustion engines which convert chemical energy of fuels directly into motion, were introduced later than electric motors and up to now, these two types of energy conversion devices are muscles of our civilization. Due to the lower cost, higher efficiency, and less environmental issues, electric motors are replacing combustion motors in most application except for where feeding with electricity is practically impossible, e.g. in airplanes. In most modern countries, the diesel locomotives which had already replaced steam locomotives in 20th century, have been replaced by electric trains.

Electric energy and invention of electric machines not only replaced steam and diesel engines in heavy industries but they brought our civilization the information age. Computers, television sets, and communication devices, to name a few, are all running on electricity. The tasks of miniature electric motors found in our computers couldn’t have been done by steam or combustion engines. More importantly electrical energy can be transmitted virtually everywhere on Earth, including deep in ocean or underground, or can be generated from solar energy in deep space to energize space satellites.
Let’s have a look at electricity consumption of our civilization. Fig. 1.1-(a) shows the global electrical energy consumption and the consumption by end-use is shown in Fig. 1.1-(b).

Total: 18.9 Trillion kWh

![Global electricity consumption in 2012](a)

![Consumption percentage of total energy by end-use in 2006.](b)

Fig. 1.1. (a) Global electricity consumption by section in 2012, (b) consumption percentage of global electricity by end-use in 2006. Source: International Energy Agency.

Although we focused our discussion of electrical machines on electric motor, there are other two categories of electrical machines; generators which are, in principle, similar to motors convert the mechanical energy into electric energy, and transformers are used as a device in power transmission and adjustment systems. We also place reactors in the third category as they have designs close to that of transformers.

Electric apparatuses cover a wider range of devices from above-mentioned electrical machines to actuators and electromagnets to as far as magnetic guns (rail guns) and magnetic levitation.
Recently, electric motors have found their application in electric cars and hybrid vehicles. Since these motors run on batteries and has to be fit in small spaces such as in the wheels or attached to the engine, high efficiency and high torque/volume ratio is required. With such diverse application and rapid development of electrical machines, optimal design becomes essential. Even a marginal improvement in efficiency of electrical machines may result in significant reduction of energy consumption and relevant environmental issues. Fig. 1.2 shows explains the impact of a marginal improvement in the global average efficiency of motors.

![Diagram showing the impact of a marginal improvement in efficiency on energy savings and nuclear reactor shutdowns.](image)

**Fig. 1.2. Impact of a marginal improvement in the average efficiency of motors**

Earlier designs were trial and error approaches based on analytical study the designer’s deep understanding of the machine principles. Several prototypes may have been built and tested prior to confirming the performance. The process is time consuming and expensive and of course competition in marketplace demands more predictive designs in shorter period. With the introduction of high speed and low cost computers, more accurate analysis of complex geometries become possible and numerical methods such as finite element method (FEM) were widely adopted for the design of electrical machines. The application of FEM reduces the time and cost of design and modifications.
As the design criteria of energy conversion machines become tighter, more accurate numerical modelling and simulation of the machines are required. As part of a worldwide effort to reduce energy consumption, CO₂ emissions and the impact of industrial operations on the environment, various regulatory authorities in many countries have introduced or are planning legislation to encourage the manufacture and use of higher efficiency motors. Fig. 1.3 shows the tightening of standard of motor efficiency within two decades.

![Efficiency Graph](image)


Permanent magnet motor with higher efficiency and higher torque/volume has found their way back in applications after the development of rare-earth magnets as the strongest type of permanent magnets made. These types of motor are often driven by inverter power supply to control the speed and also the starting current.
In addition to the energy standards, environmental noise standards of motors are also becoming tighter. As motors may run continuously in the same space as humans does, the design of low-noise machines become very important. Acoustic noise of machines is originated from either mechanical imperfectness such as bearing defects of rotor eccentricity or from the magnetic forces as well as a phenomenon known as magnetostriction. Noise calculation of electrical machines is an interdisciplinary field of research including magnetic field analysis, structural dynamic analysis and acoustic analysis. After such a long introduction, it is the time to discuss the research background and motivation in the following sections.

1.1 Research Background

Electrical machines such as motors and generators are designed to convert electric/mechanical energy into mechanical/electric energy respectively. Other electrical devices such as actuators, electromagnets, and electromechanical relays are designed to cause a desired displacement. In both classes of devices, motion often occurs due magnetic force caused by magnetic field in the device. On the other hand, devices such as transformers and reactors are not designed to create motion but to improve electrical characteristics of the device, yet due to magnetic field, magnetic force is present and the force causes small undesired deformation or vibration. The small displacements due to forces of magnetic origin or any other sources may be negligible but they can cause significant mechanical stress in the iron core and the
stress could in turn, cause a significant change in the magnetic property of the material which should be considered in the design of such devices [13].

Fig. 1.4. Flowchart of interactions between magnetic and mechanical quantities in magneto-mechanical problems.

1.1.1 Considering Mechanical Stress in the Analysis of Electromagnetic Devices

Mechanical stress can modify the magnetic permeability of ferromagnetic materials through a phenomenon known as the inverse magnetostriction effect (IME) [13]. It also affects the magnetostriction (MS) characteristic of the material [23]. Magnetostriction contributes to the deformation and vibration of electrical machines and apparatuses [24] whereas IME can modify the magnetic flux density distribution through modifying the permeability. Therefore, when the induced stress is relatively high, accurate calculation of the flux density and consequently the loss and vibration requires a coupled analysis of the problem which takes account of the dependence of magnetic and magnetostriction characteristics on the stress.
In addition to magnetically induces stress, there magnetic media might be under mechanical stress due to fitting or punching. In particular, the shrink-fit technique which is used to place the motor stator in the motor housing firmly, induces high compressive stress on the core which can deteriorate the magnetic property and increase the iron loss. Moreover, the vibration of devices can cause intolerable environmental noise and this is a big concern when humans and the device share the same space. Vibration analysis of devices also demands structural dynamic analysis. Thus the study of electrical machines and apparatuses is not complete without combining mechanical and magnetic field analyses.

Due to complexity of device geometry, magnetic and mechanical analysis has to be done numerically and the finite element method (FEM) is the most appropriate method of choice [4]. Therefore, analysis and design of electrical machines demand coupled magneto-mechanical analysis using finite element method.
1.1.2 Vibration Analysis of Permanent Magnet (PM) Motor Taking Account of Magnetostriction

In ferromagnetic materials such as iron, there exists a phenomenon called magnetostriction (MS). Magnetostriction is the property of magnetic materials which manifests as change of dimension in presence of magnetic field [21]. This change of dimension is different from the deformation caused by the familiar magnetic forces. Although both deformation have the same nature, i.e. caused by magnetic forces in microscopic scale, in macroscopic Maxwell equations which are the governing equations of electrical devices, the microscopic magnetic forces cannot be taken into account. The effect of these forces appears as deformations in the macroscopic scale. Magnetostriction strain of the sample of material is often measured versus various levels of flux density and the measured data can be used in the analysis of electrical devices to obtain the deformation of the whole structure due to MS as well as to the classical magnetic forces [21].

![Illustration of magnetostriction phenomena in a two dimensions.](image)

Fig. 1.6 Illustration of magnetostriction phenomena in a two dimensions.
Electric motors, which may be running in our environment continuously, should be designed to have low acoustic noise because the consequence of continuous high noise on human psychology could turn very serious.

![Fig. 1.7 Mechanical structure of a permanent magnet motor.](image)

Permanent Magnet (PM) motor has been proposed as a strong candidate of the driving source of electric vehicles, not only because of its high torque output and high efficiency, but also because they are easily controlled over a wide range of speeds [1]. It is expected that the development of electric machines, mostly PM machines and associated power electronics in the next few years will be stimulated by large scale applications such as computer hardware, residential and public applications, land, sea and air transportation and renewable energy generation [1]. In the last two decades new topologies of high torque density PM motors, high speed PM motors, integrated PM motor drives, and special PM motors have gained maturity. The largest PM brushless motor in the world rated at 36.5 MW, 127 rpm was built in 2006 by DRS Technologies, Parsippany, NJ, U.S.A. The use of PM brushless motors has become a more attractive option than induction motors. Rare
earth PMs can not only improve the motors steady-state performance but also the power density (output power-to-mass ratio), dynamic performance, and quality. The prices of rare earth magnets are also dropping, which is making these motors more popular.

In general, rotary PM motors for continuous operation are classified into:

- d.c. brush commutator motors
- d.c. brushless motors
- a.c. synchronous motors

We are concerned with a.c. synchronous motors which have found application as the drives in electric cars. These motors are fed with three phase voltage and are often controlled by an SPWM inverter. Vibration and acoustic noises in such motor are considered a critical issue as the consumer is continuously and directly affected by the acoustic noise. There are several electromagnetic sources that affect the vibration of the motor: cogging torque, radial force, torque ripple, etc. These sources, once represented as forces, have specific harmonics. These harmonics, if fall close to natural frequency of the device structure, can cause large vibrations. Therefore, by computing natural frequencies of the device and the harmonics of such sources at the design stage, a considerable reduction in the vibration and acoustic noises can be achieved. Computation of natural frequencies of structures and determination of the participation of each mode on the total vibration of the structure is known as the modal analysis [32] which requires solving generalized eigenvalue problems with often very large matrices. Solving such problems are quite challenging and demands high amount of computer memory and computation time.
The vibration analysis of structures including those of electrical machine requires solving motion equation with large degrees of freedom. In case of devices powered with inverter power supplies, due to the high switching frequency of the inverter, small time steps are required to capture the vibration characteristics of the device. In addition to the study of natural frequencies and modes of vibration of the structure, the modal analysis method is also needed in modal decomposition method in solving motion equation. It is often common to consider a limited number of lowest natural frequencies in this method but while is some cases a limited number of modes gives very good approximations of the solution, in some problems the number of modes required to for a desired approximation is not known in priori. This is a serious issue because one may obtain totally wrong results. A details study on the solution methods of motion equation is also required to make sure the equation is solved efficiently and correctly.

1.2 Research Purposes

1.2.1 Coupled Magneto-Mechanical Analysis

Several models have been proposed for the analysis of magneto-mechanical problems taking account of significant coupling terms in certain problems. To mention a few, in [18] and [19] IME is taken account of through measured permeability curves dependent on stress while MS is neglected. In [20] beside IME, MS is also poorly considered. In [21] IME is treated for the certain case of the body clamped on all sides. General models taking account of MS as well as IME are
proposed in [22] and [23] but the models assume MS to be independent of stress. A novel model which relies on a number of fitting parameters rather than magnetization and MS curves is proposed in [24], describing MS dependence on magnetic field and stress but it is unsuccessful in modeling magnetization behavior under stress [25]. In this thesis, motivated by the successful modeling of magnetostriction in [25], we propose a general model for solving coupled magneto-mechanical problems in isotropic materials. The model which takes account of MS and IME through a set of curves describing MS as a function of flux density and stress as in [25], and a single stress-free magnetization curve. The model initially assumes that the B and H fields remain parallel even under stress. The assumption is valid if the stress is dominantly applied along the direction of the magnetic flux density, which is often the case with the magnetically induces stress. The assumption however fails if there exists applied stress in the transverse direction. We further extend the method to consider the coupling under multi-axial stress. The proposed method are based on the energy balance between magnetic and mechanical stored energy in the structure.

1.2.2 Vibration Analysis of Electric Motors

Vibration analysis of PM motors has been an emerging topic of research in recent years as a result of the increased application of IPM motors in home appliances as well as electric vehicles [47], facing tighter criteria of low acoustic noise emission. Some authors have investigated the vibration characteristics of IPM motor [21] using the finite element method [17] (FEM), while others used less accurate analytical methods [42], but in their works, the magnetostriction (MS) is not considered. The
effect of the MS on the vibration of induction motor has been thoroughly investigated in 2-D [21], [41]. However, to our knowledge, vibration characteristics of IPM motor considering MS and with voltage-input has not been reported yet. A particular challenge in the analysis of IPM motors is that the input voltage usually has sinusoidal pulse width modulation (SPWM) waveform with high carrier frequency which requires very small time steps for the calculation of the magnetic field and the currents. These harmonics can affect the vibration of the motor through magnetic and MS forces. Some researchers have evaded the challenge of current calculation by using measured currents in the analysis, but for a motor in the design stage, this option is not available. In this study, a 2-D magnetic field analysis of an 8-pole 36-slot IPM motor with SPWM input voltage is carried out and the coil currents are calculated. Next, the effect of the MS on the vibration characteristics of the motor is investigated using 3-D finite element structural dynamic analysis.
1.3 Organization of Thesis

This thesis tries to cover the above mentioned topics from the numerical point of view, and it is organized as follows:

Chapter 1 is the introduction to the thesis. It explains the research purpose, and the background.

In Chapter 2, the Maxwell’s equations are explained and the governing equations of magnetic field analysis coupled with circuits are derived. Then the finite element discretization in two and three dimensions using Galerkin method is discussed. Iterative matrix solvers for solving the system of finite element equations are introduced and finally the magnetic force calculation is explained. Afterwards, magnetostriction which is one of the key topics of this thesis is discussed and magnetostriction force calculation is explained. Finally the conventional core loss calculation method as well as a proposed method of core loss calculation taking account of stress are explained.

Chapter 3 discusses static structural analysis and mechanical stress calculations. The chapter includes the calculation of the stress due to thermal contraction which is usually required to calculate the shrink-fit stress in the stator.

Chapter 4 is dedicated to the coupled magneto-mechanical problems and the relevant solution methods.

In Chapter 5 structural dynamic analysis which leads to motion equation is discussed. Then the solution methods of motion equation for the vibration analysis is
explained and finally numerical results are compared by applying them to two simple problems.

In Chapter 6, the application of the magneto-mechanical analysis in the vibration analysis of an IPM motor is discussed. The magnetic field analysis results are presented first and then the core loss calculation are compared for two cases of neglecting and considering the shrink-fit stress. Next, the vibration analysis is carried out for three cases; MS and stress neglected, MS considered and stress neglected, and both MS and stress considered. In addition a comparison between the solution methods of motion equation is made to investigate the possibility of reducing the computational costs of the vibration analysis.

Finally, Chapter 7 draws conclusions and sets forward recommendations on the methods.
Chapter 2 Magnetic Field Analysis

2.1 Introduction

From ancient times, electricity and magnetism have been described only qualitatively and nobody had any clue about their close connection. Later on, after Newton's big success with the mathematical description of mechanics, many people started to think about the electric forces and the magnetic forces and made models very similar to Newton's principle law governing the gravitational.

In the early 19th century, when batteries were already available and to establish current in a circuit. Oersted incidentally observed that around a current-carrying wire, compass needles are moving similar to the same effect caused by a permanent magnet of the Earth's magnetic field [2]. Then again other physicists such as Weber and Ampere made models to describe this phenomenon, and the models in turn were checked with experiments and showed agreement. Faraday came up with a new idea to describe electromagnetic phenomena, namely the idea that there are not actions at a distance or instantaneous forces caused by electric charges and currents but that these sources are causes for electric and magnetic fields, and that these fields in turn are the cause of the forces of charged or magnetized bodies at their position in this field. In this way the picture of interactions became localized, i.e., the cause of forces is fields at the point where the body is located at this moment in time. Faraday also observed that magnetic fields, changing with time can cause electric fields curling around the changing magnetic fields (Faraday's law of induction), and Maxwell finally came up with a consistent model for the dynamics of the fields and the charge
and current distributions causing them and how the fields mediate the electric and magnetic forces of charged and magnetized bodies [2].

At one point, when combining a lot of observations, he came to the conclusion that Ampere's law, how to describe the magnetic fields as caused by electric current had some important flaw, and that he had also to assume that a time-varying electric field must also cause magnetic fields, very similar to electric currents. Thus Maxwell interpreted a time-varying electric field as a "displacement current", causing magnetic fields in the very same way as electric currents. This is called the Ampere-Maxwell Law. Also Faraday's Law of induction, of course, had been incorporated into Maxwell's beautiful equations. Combining all these finding we, in general, have time-varying magnetic fields that cause time-varying electric fields and these time-varying electric fields in turn cause magnetic fields. It turns out in this way that electric and magnetic fields make up one coupled set of quantities, nowadays called "the electromagnetic field", and that this field has its own existence as a dynamical entity in its own right as much as material bodies have their existence in everyday life.
2.2 Maxwell Equations

The electromagnetic field follows the four Maxwell equations in differential form as follows [5-6]:

\[ \nabla \times H = J \frac{\partial D}{\partial t} \]  
(2-1)

\[ \nabla \times E = -\frac{\partial B}{\partial t} \]  
(2-2)

\[ \nabla \cdot D = \rho \]  
(2-3)

\[ \nabla \cdot B = 0 \]  
(2-4)

where \( B, H, J, D, E \) are the magnetic flux density, magnetic field intensity, \( \nabla \times (\cdot) \) and \( \nabla \cdot (\cdot) \) denote the rotation and divergence operators, respectively.

The constitutive relations, which define the relationship between the field quantities are:

\[ B = \mu H , \]  
(2-5)

\[ J = \sigma E , \]  
(2-6)

\[ D = \varepsilon E , \]  
(2-7)

where \( \mu, \sigma, \) and \( \varepsilon \) are the permeability, the conductivity and the permittivity, respectively.

In electrical machines and magnetic devices, the displacement current in Eq. (2-1) is negligible as the operational frequency is relatively low. This is referred to as quasi-static case. So Eq. (2-1) becomes:

\[ \nabla \times H = J \]  
(2-8)
In this case, the displacement current $D$ is removed from the Maxwell equations and the following equation for the continuity of current (derived by taking the rotation of Eq. (2-1)) is considered instead of Eq. (2-3).

$$\nabla \cdot J = 0$$

(2-9)

Eqs. (2-1), (2-2), (2-4), and (2-9) can be combined altogether into two equations known as $A-\phi$ formulation explained in the following section.

### 2.3 $A-\phi$ Magneto-Dynamic Formulation of Maxwell’s Equations

Using the vector identity that “divergence of rotation of any continuous vector field is zero”, Eq. (2-4) suggests that $B$ is the rotation of a continuous field [8], [59]:

$$B = \nabla \times A.$$  

(2-10)

The vector field $A$ is called the magnetic vector potential. It guarantees that the Eq. (2-4) is always satisfied if the flux density $B$ is expressed in terms of an auxiliary continuous vector field $A$.

The current density $J$ in Eq. (2-8) is the sum of the exciting current density $J_0$ and eddy current (induced current) $J_e$, so Eq. (2-8) becomes:

$$\nabla \times H = J_0 + J_e.$$  

(2-11)

Substitute Eqs. (2-5) and (2-10) in Eq. (2-11), it becomes:

$$\nabla \times (\nabla \times A) = J_0 + J_e.$$  

(2-12)
where $\nu$ is the magnetic reluctivity, i.e. the inverse of the magnetic permeability.

Substituting Eq. (2-5) in Eq. (2-2) yields:

$$\nabla \times \left( E + \frac{\partial A}{\partial t} \right) = 0$$

(2-13)

Since the rotation of gradient of any continuous scalar field is zero, $E$ is expressed as:

$$E = -\frac{\partial A}{\partial t} - \nabla \phi,$$

(2-14)

where $\phi$ is electric scalar potential and $\nabla(\cdot)$ denotes gradient operator. Eq. (2-14) tells us that the scalar potential only describes the conservative electric field generated by electric charges. The electric field induced by time-varying magnetic fields is non-conservative, and is described by the magnetic vector potential $A$.

Using Eqs. (2-6) and (2-15), Eq. (2-12) becomes

$$\nabla \times \left( \nu \nabla \times A \right) = J_0 - \sigma \left( \frac{\partial A}{\partial t} + \nabla \phi \right).$$

(2-15)

The second term on the right-hand-side of (2-15) is the *eddy current* density. Since the excitation current is definitely divergence free, According to Eq. (2-6) the eddy current should also be divergence free:

$$\nabla \cdot \left\{ -\sigma \left( \frac{\partial A}{\partial t} + \nabla \phi \right) \right\} = 0.$$

(2-16)

Eqs. (2-15) and (2-16) represents the $A$-$\phi$ formulation of magneto-dynamic Maxwell’s equations. In non-conductive regions $\sigma = 0$, so $\phi$ and also Eq. (2-16) becomes irrelevant and the formulation takes the following simple form.

$$\nabla \times \left( \nu \nabla \times A \right) = J_0.$$

(2-17)
Reviewing Eqs. (2-13) to (2-16) reveals that for any arbitrary \( \phi \) there is an \( A \) which satisfies all the equations. For example, by setting \( \nabla \phi = 0 \), the \( A \cdot \phi \) formulation reduces to the \( A \) formulation (\( A \)-method) which also gives the same \( B \) and eddy current but the \( A \cdot \phi \) formulation turns to be advantageous from numerical points of view [60].

2.4 Field-Circuit Equations

Solving equations in the previous section requires the knowledge of \( J_0 \). Most electrical devices are powered through windings made up of tens or hundreds of turns of wire. If the current of the wire is known, the current density can be approximated in term of the current according to the following equations:

\[
J_0 = \frac{NI}{S} \hat{t},
\]

where \( N \) is number of turns, \( I \) is the current passing through a single strand of wire, \( S \) is the coil cross-section area and \( \hat{t} \) is a unit vector normal to the surface and pointing
to the direction that the current flows. However, electrical devices are often fed by a voltage source rather than a current source and the current cannot be calculated simply from voltage without solving the field equation. Fig. 2.2 shows the schematic diagram of field-circuit coupling.

![Diagram of field-circuit coupling](image)

**Fig. 2.2. Diagram of field-circuit coupling.**

At a given cross-section $S$ of the coil, the same induced current flows in all loops of wires which are in series and if the wires cross-sections are equal, thus the eddy current densities are also equal. The eddy current density in each wire becomes

$$J_k^e = -\sigma k \left( \frac{\partial A}{\partial t} + \nabla \phi \right)^k$$

where $k$ is the wire index. If, in addition, $\phi$ are equal, the sums within the brackets for all wire become equal as well. Thus if the coil is densely wound, it is justified to approximate the potentials at each point on the cross-section with average vector potential $A_{ave}$, and average scalar electric potential $\phi_{ave}$ over the coil cross-section:

$$A_{ave} = \frac{1}{S} \int_A ds , \quad \phi_{ave} = \frac{1}{S} \int \phi ds$$

Thus, there is an average electric field as follows:

$$E_{ave} = -\frac{\partial A_{ave}}{\partial t} - \nabla \phi_{ave}$$
Integrating along one loop of the wire gives the induced electromotive force along the loop. The total induced electromotive force $EMF$ then becomes:

$$EMF = N \oint \mathbf{E} \cdot d\mathbf{l} = -N \oint \frac{\partial \mathbf{A}}{\partial t} \cdot d\mathbf{l} - N \oint \nabla \phi \cdot d\mathbf{l}$$

The second integral on the right-hand-side vanishes because the line integral of any gradient field over a closed path is zero. The first integral can be written as follows:

$$EMF = -\frac{N}{S} \int \frac{\partial \mathbf{A}}{\partial t} \cdot d\mathbf{S} \cdot d\mathbf{l} = -\frac{N}{S} \int \frac{\partial \mathbf{A}}{\partial t} \hat{\mathbf{t}} d\mathbf{V}$$

where $\hat{\mathbf{t}}$ is a unit vector in the direction of current flow. Now according to the circuit law:

$$L \frac{dI}{dt} + RI + EMF = V,$$

where $L$ is the sum of the external and leakage inductances and $R$ is total resistance in the current path. Thus, the coupled circuit equations become

$$\nabla \times (\nabla \times \mathbf{A}) = \begin{cases} \frac{NI}{S} \hat{\mathbf{t}} & \text{in stranded coils} \\ J_0 - \sigma \left( \frac{\partial \mathbf{A}}{\partial t} + \nabla \phi \right) & \text{else} \end{cases} \tag{2-23}$$

$$L \frac{dI}{dt} + RI - \frac{N}{S} \int \frac{\partial \mathbf{A}}{\partial t} \hat{\mathbf{t}} d\mathbf{V} = V \tag{2-24}$$

In (2-23), $J_0$ the exciting current density in solid conductors if any. If all the coils are coupled with voltage sources, $J_0$ is zero.
2.5 Finite Element Discretization of Differential Equations

It is simpler to explain the finite element method in the context of first order nodal elements. In this method, the domain of analysis is first discretized into a number of smaller volumes known as elements. Within each element, the continuous field, here $A$, or the continuous scalar filed, here $\phi$, is considered unknown at element vertices. The field everywhere inside the elements is approximated by interpolating over the elements using interpolating functions more commonly known as shape functions. In this section the discretization is demonstrated in 3-D space and with hexahedral elements although there are a variety of element types to be chosen [46]. Tetrahedral elements are perhaps the most common type for 3-D discretization because the automatic mesh generation with tetrahedrons is simpler. Other types are pentahedrons and wedge elements. In 2-D space, triangular or quadrilateral elements are the most common while in 3-D discretization, tetrahedral and hexahedral elements most used. The hexahedral elements have a better accuracy but automatic generation is not as simple as the tetrahedral element.
Fig. 2.3. Some types of elements used in finite element discretization, (a) 2-D elements, (b) 3-D elements.

Fig. 2.4 Discretization of 3-D field $A$ by hexahedral elements. The unknowns are the field values at the nodes.

Thus, the field in the middle element of Fig. 2.4 is interpolated as follows [10-12]:

\[
\begin{array}{c}
\vdots \\
A_1 & A_2 & A_3 & A_4 \\
\vdots \\
\end{array}
\]
The continuous interpolating functions $N_j$ (also known as shape functions) are such that their values are 1.0 at the corresponding node decay to zero at others nodes. Moreover $N_j$ is zero inside all elements which do not share the corresponding node.

It is common to write (2-25) in the following form:

$$A = \sum_{j=1}^{8} N_j A_j \quad (2-26)$$

However the subscript $j$ here has a different notation. It refers to the so-called local numbering of the fields and the shape functions. It is obvious that if all the nodal values are equal, the interpolated field inside the element is also equal to the nodal values, thus

$$\sum_{j=1}^{8} N_j = 1 \quad (2-27)$$

The interpolating functions depend on the element type, and they are not unique for a given element type either. After defining the shape functions and the discretized space, the weighted residual Galerkin method is used to obtain a set of linear equations whose solution are the unknown values of the field. The method is explained in the following section.

2.5.1 Weighted Residual Galerkin Method

The weighted residual method transforms the original differential equation into its weak form. The method is explained by applying it to (2-17) [10-12]:

$$A_{x2} = N_2 A_2 + N_3 A_3 + N_6 A_6 + N_7 A_7 + N_10 A_{10} + N_{11} A_{11} + N_{14} A_{14} + N_{15} A_{15} \quad (2-25)$$
\[ \mathbf{G}_i = \int_V W_i \left\{ \nabla \times (\nu \nabla \times \mathbf{A}) - \mathbf{J}_o \right\} dV = \mathbf{0}, \quad (2-28) \]

where \( \mathbf{G}_i \), being a vector, is referred to as the weighted residual and \( V \) is the entire domain. As we can see, the differential equation has been multiplied by some weight functions \( W_i \) (associated with unknown \( i \)) and then integration over the entire domain of analysis is set to zero. The above equations are called the weak formulation. According to the fundamental lemma of calculus of variation, if Eq. (2-28) holds for every \( W_i \), the integrant is identical to zero. If instead of every possible \( W_i \), a limited number of weight function is chosen, they are called trial functions and the integrant may not be exactly but only approximately zero. By choosing \( W_i = N_i \) (are the same function as in Eq. (2-25)), the method is called the weighted residual Galerkin method.

Before breaking the entire integration into integrations over elements, Eq. (2-28) can be simplified. Recalling following vector identity:

\[ \nabla \times (\psi \mathbf{B}) = \psi \nabla \times \mathbf{B} + \nabla \psi \times \mathbf{B}. \quad (2-29) \]

The first term in Eq. (2-28) can be simplified as follows:

\[ \psi = N_i, \quad \mathbf{B} = \nabla \times \mathbf{A}, \]

\[ \int_V N_i \left\{ \nabla \times (\nu \nabla \times \mathbf{A}) \right\} dV = \int_V \nabla \times (N_i \nu \nabla \times \mathbf{A}) dV - \int_V \nabla N_i \times (\nu \nabla \times \mathbf{A}) dV \]

\[ = \int_S N_i \nu \nabla \times \mathbf{A} \times dS - \int_V \nabla N_i \times (\nu \nabla \times \mathbf{A}) dV, \]

where \( S \) includes the element interfaces as well as the boundary of \( V \). Generalized Stokes' theorem was used to convert the first volume integral on the right-hand-side into a surface integral. On the domain boundary, the magnetic field \( \mathbf{B} \) is either zero or
known, normal to the surface in case of Neumann boundary condition, so the surface 
integrals associated with the domain boundary. As for the element boundary, 
since $\mathbf{v} \nabla \times \mathbf{A} = \mathbf{H}$, and the tangential component of $\mathbf{H}$ is continuous, vanishes if $\mathbf{A}$ is 
the exact solution. Since the finite element gives the approximation of $\mathbf{A}$, by setting 
the integral to zero, the boundary conditions mentioned above is weakly imposed in 
the formulation, so in this method the tangential component of $\mathbf{H}$ is not continuous 
exactly. However the boundary condition of $\mathbf{B}$ is satisfied exactly, i.e. the normal 
component of $\mathbf{B}$ is exactly continuous due to the continuity of the solution, $\mathbf{A}$, on the 
element boundaries.

Within each elements can also write:

$$
\nabla \times \mathbf{A} = \nabla \times \left( \sum_{j=1}^{N_{el}} N_j \mathbf{A}_j \right) = \sum_{j=1}^{N_{el}} \nabla N_j \times \mathbf{A}_j . 
$$

(2-30)

Hence

$$
- \sum_{i=1}^{N_{el}} \sum_{j=1}^{N_{un}} \int_{V_e} \nabla N_i \times (\mathbf{v} \nabla N_j \times \mathbf{A}_j) dV = \sum_{i=1}^{N_{el}} \int_{V_e} \mathbf{N} \cdot \mathbf{J}_0 \ dV , \quad i = 1, 2, \ldots, N_{un} ,
$$

(2-31)

where $N_{el}$ and $N_{un}$ are number of elements and number of unknown nodal values, 
respectively and $V_e$ denotes element volume. The exciting current $\mathbf{J}_0$ in the element is 
expressed as follows by using the vector. The unknowns are $\mathbf{A}_j$. Eqs. (2-31) can be 
written in the block matrix form as follows:

$$
\begin{pmatrix}
G_{11} & G_{12} & \cdots & G_{1N_{un}} \\
G_{21} & G_{22} & \cdots & G_{2N_{un}} \\
\vdots & \vdots & \ddots & \vdots \\
G_{N_{el}1} & G_{N_{el}2} & \cdots & G_{N_{el}N_{un}}
\end{pmatrix}
\begin{bmatrix}
\mathbf{A}_1 \\
\mathbf{A}_2 \\
\vdots \\
\mathbf{A}_{N_{un}}
\end{bmatrix}
= 
\begin{bmatrix}
\mathbf{b}_1 \\
\mathbf{b}_2 \\
\vdots \\
\mathbf{b}_{N_{un}}
\end{bmatrix},
$$

(2-32)
where $G_{ij}$ are 2x2 or 3x3 matrices in 2-D or 3-D spaces, respectively given by the following equation:

$$G_{ij} = -\sum_{i=1}^{N_e} \int_{V_e} H_{ij} dV,$$  \hspace{1cm} (2-33)

where $H_{ij}$ is some matrix which satisfies the following equation:

$$\nabla N_i \times (\nabla N_j \times A_j) = H_{ij} A_j,$$ \hspace{1cm} (2-34)

and $b_i$ is given as follows:

$$b_i = \begin{bmatrix} b_{i,x} \\ b_{i,y} \\ b_{i,z} \end{bmatrix} = \sum_{i=1}^{N_e} \int_{V_e} N_i \cdot J_0 \ dV.$$ \hspace{1cm} (2-35)

Eq. (2-32) is often written in a compact form as follows:

$$KA = b$$ \hspace{1cm} (2-36)

Were $K$ is symmetric positive-definite matrix whose dimension is the number of unknowns. It is often referred to as the reluctance or magnetic stiffness matrix. By solving Eq. (2-36) $A$ is found for all unknowns then $B$ can be easily calculated as the rotation of $A$ and using Eq. (2-30). However, knowledge of $N_i$ is required for computation of $H$ and $b$. In addition, the integrations in Eqs. (2-33) and (2-35) should be carried our numerically and efficiently. Both issues depend on the type of elements. We choose the hexahedral element type as the most general and most accurate one for 3-D discretization.
For easy definition of shape functions $N_i$ and also easy implementation of numerical integration, the element is first transformed to a local coordinate system in the element becomes a cube with sides of 2.0 unit length centered at the origin. This element is referred to as the reference element.

Fig. 2.5. First-order hexahedral element. (a) Global coordinate system $(x, y, z)$, (b) local coordinate system $(\xi, \eta, \zeta)$.

It is easy to verify that the following first-order functions in the local coordinate system meet the requirement of shape functions mentioned earlier:

$$N_i = \frac{1}{8} \left( 1 + \xi_i \xi \right) \left( 1 + \eta_i \eta \right) \left( 1 + \zeta_i \zeta \right) \quad (i = 1\sim8) \quad (3-37)$$

where the local coordinates $\xi_i, \eta_i, \zeta_i$ of the nodes are given in the TABLE 2.1.
The relation between the global and local coordinate systems is given by:

\[
\begin{align*}
\sum_{i=1}^{8} N_i x_i, & \quad y = \sum_{i=1}^{8} N_i y_i, \quad z = \sum_{i=1}^{8} N_i z_i. \quad (2-38)
\end{align*}
\]

The gradients in Eq. (2-34) should also be transformed into the local coordinate system, it can be shown that:

\[
\nabla N_i = J^{-1} \begin{bmatrix}
\frac{\partial N_i}{\partial \xi} \\
\frac{\partial N_i}{\partial \eta} \\
\frac{\partial N_i}{\partial \zeta}
\end{bmatrix}, \quad (2-39)
\]

Where \( J \) is the Jacobian matrix expressed as:

\[
J = \begin{bmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\
\frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta}
\end{bmatrix}. \quad (2-40)
\]
Therefore the integration expressed by $x$, $y$, $z$ in the global coordinate system, for example the $xx$-component $K_{ij}$ in (2-33), corresponds to that expressed by $\xi$, $\eta$, $\zeta$ in the local coordinate system as follows:

$$
\int \int \int \int \int K_{ij,x} \ dV = \int \int \int K_{ij,x} (\xi, \eta, \zeta) |J| d\xi d\eta d\zeta ,
$$

(2-40)

where $|J|$ is determinant of $J$. The integration can be efficiently performed using the Gauss integration rule in each dimension. In this method, the integrals are replaced by summations of weighted values of functions at specific evaluation points [66]. In one dimension,

$$
\int f(x) \ dx = \sum_{k=1}^{n} w_k f(x_k).
$$

(2-41)

TABLE 2.2 shows the integration points, $x_k$, and weights, $w_k$, for the Gauss integration rule. With $n$ points, the method gives exact integration for integrands of orders up to $2n-1$ and for higher orders, with some error.

The integration in (2-40) becomes

$$
\int K_{ij,x} \ dV = \sum_{k_x} \sum_{k_y} \sum_{k_z} w_{k_x} w_{k_y} w_{k_z} K_{ij,x} (\xi_{k_x}, \eta_{k_y}, \zeta_{k_z}) |J(\xi_{k_x}, \eta_{k_y}, \zeta_{k_z})|.
$$

(2-42)

It is easy to verify that for hexahedral elements, the determinant of the Jacobean matrix and also $K_{ij}$ are at most second-order in each dimension, therefore whole integrand is fourth-order, so if $\nu$ is constant within the element as in linear materials, $n=3$ is enough to calculate the integral exactly. In rectangular prism elements with edges parallel with the coordinate axes, the determinant becomes a constant, so $n=2$
gives the exact integration. In nonlinear materials \( n=3 \) often gives very good approximation of the integral.

**TABLE 2.2. Integration points and corresponding weights of the Gauss integration rule [66].**

<table>
<thead>
<tr>
<th>Number of points, ( n )</th>
<th>Point, ( x_k )</th>
<th>Weight, ( w_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>(- p, p ) ( p = \sqrt{\frac{1}{3}} )</td>
<td>1,1</td>
</tr>
<tr>
<td>3</td>
<td>(- p, 0, p ) ( p = \sqrt{\frac{3}{5}} )</td>
<td>( \frac{5}{9}, \frac{8}{9}, \frac{5}{9} )</td>
</tr>
<tr>
<td>4</td>
<td>(- p, -q, 0, q, p ) ( p = \sqrt{\frac{3}{7} + \frac{2}{7} \sqrt{\frac{6}{5}}} ), ( q = \sqrt{\frac{3}{7} - \frac{2}{7} \sqrt{\frac{6}{5}}} )</td>
<td>( w_p, w_q, w_q, w_p ) ( w_p = \frac{18 - \sqrt{30}}{36} ), ( w_q = \frac{18 + \sqrt{30}}{36} )</td>
</tr>
</tbody>
</table>

Eq. (2-30) is recalled with distinction between the linear and nonlinear material property:

**linear material** \( KA = b \) \hspace{1cm} (2-43-a)

**nonlinear material** \( K(A)A = b \) \hspace{1cm} (2-43-b)

For nonlinear materials, \( v \) depends on \( B \) and thus on \( A \). Therefore the coefficient matrix \( K \) depends on \( A \) and the equation is nonlinear. For linear case the equation can
be solved in one step using a proper method which shall be explained later. Nonlinear system of equations can be solved using Newton-Raphson method. In this method starts with and initial, at each step by linearizing around the previous solution, a next solution is found which is often closer to the exact solution.

\[
K(A)A = K(A_k)A_k + \frac{\partial}{\partial A}(K(A)A)dA
\]

\[
= K(A_k)A_k + (K(A) + \frac{\partial K(A_k)}{\partial A}A_k)dA
\]  

(2-44)

Thus

\[
(K(A_k) + K'_k)dA = b - K(A_k)A_k
\]

(2-45)

\[
K'_k = \frac{\partial K(A_k)}{\partial A}A_k
\]

(2-46)

The solution is improved to \(A_{k+1} = A_k + dA\). The procedure is repeated until \(dA\) or the difference between the two successive \(B\) becomes small enough.
2.5.2 Three-Dimensional Discretization Using Hexahedral Edge Elements

Although in this thesis, no 3-D magnetic field analysis is carried out, the discretization of the fundamental equations of \( A - \phi \) method, i.e. Eqs. (2-15) and (2-16) in 3-D space using hexahedral edge elements is explained due to its importance in this field. Traditionally node elements described above were used in the magnetic field analysis. Although in 2-D these type of elements are still used, with the introduction of edge elements in 3-D finite element, in which the unknowns are the rotation of field along the sides of elements, the node elements in 3-D magneto-dynamic analysis were outdated as the edge elements offer better accuracy.

The first-order hexahedral element with twelve vector edges in the local coordinate system \((\xi, \psi, \zeta)\) is shown in Fig. 2.6 [5-12].

![First-order hexahedral edge element with numbered unknown edges vector.](image)

**TABLE 2.3 Local coordinates for edges.**

<table>
<thead>
<tr>
<th>Edge number ( k )</th>
<th>( \xi )</th>
<th>( \eta )</th>
<th>( \zeta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>9</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>11</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>12</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>
The vector shape functions \( N^{:ed}_{j} \) for edge \( j \) in the edge hexahedral finite element is defined in the local coordinate system and expressed as follows:

\[
N^{:ed}_{j} = \begin{cases} 
\frac{1}{8} (1 + \xi_j \eta_j) (1 + \zeta_j \xi_j) \nabla \xi & (j = 1 \sim 4) \\
\frac{1}{8} (1 + \zeta_j \xi_j) (1 + \xi_j \zeta_j) \nabla \eta & (j = 5 \sim 8) \\
\frac{1}{8} (1 + \xi_j \zeta_j) (1 + \eta_j \xi_j) \nabla \zeta & (j = 9 \sim 12)
\end{cases}
\] (2-47)

The local coordinates \( \xi_j, \eta_j, \zeta_j \) of the edges are given in TABLE 2.3.

The scalar \( \phi \) in one hexahedral element for the magnetic field analysis is obtained using the \( \phi_i \) on each node of the element as follows:

\[
\phi = \sum_{i=1}^{N} N_i \phi_i
\] (2-48)

The vector \( A \) within the element is obtained using the \( A_j \) on each edge of the element as follows:

\[
A = \sum_{j=1}^{12} N^{:ed}_{j} A_j
\] (2-49)

Note that unlike the nodal shape function, \( N^{:ed}_{j} \) has the unit of \( \text{m}^{-1} \). On the other hand, the unknown edge vectors are the line integral of \( A \) along the edge, i.e. the product with the edge length, so the product has the S.I. unit of magnetic vector potential.

By applying the weighted residual Galerkin method, with vector shape functions \( N^{:ed}_{j} \) as test functions, to Eqs. (2-15) and (2-16), the following residual equations can be obtained [28-30]:
\[ G_i = \int_V N_i^{ed} \cdot \left( \nabla \times (\nu \nabla \times A) - J_\phi + \sigma \left( \frac{\partial A}{\partial t} + \nabla \phi \right) \right) dV = 0, \quad (2-50) \]

\[ G_{di} = \int_V N_i^{ed} \nabla \cdot \left( \sigma \left( \frac{\partial A}{\partial t} + \nabla \phi \right) \right) dV = 0. \quad (2-51) \]

\( G_i \) and \( G_{di} \), being scalars, are the residuals corresponding to Eqs. (2-15) and (2-16), respectively. The first term in Eq. (2-50) can be changed as follows:

\[ \int_V N_i^{ed} \cdot \nabla \times (\nu \nabla \times A) dV = \int_V \nabla \cdot (\nu \nabla \times A \times N_i^{ed}) dV + \int_V \nu \nabla \times A \cdot \nabla \times N_i^{ed} dV \]

\[ = \int_S \nu \nabla \times A \times N_i^{ed} \cdot n dS + \int_V \nu \nabla \times A \cdot \nabla \times N_i^{ed} dV \]

\[ = \int_S (\nu \nabla \times A \times n) \cdot N_i^{ed} dS + \int_V \nu \nabla \times A \cdot \nabla \times N_i^{ed} dV \quad (2-52) \]

where \( S \) is the boundary of \( V \), \( n \) is the outward normal vector of \( S \). In the proceeding above, the following vector operations and the Gauss theorem are used:

\[ \nabla \cdot (A \times B) = B \cdot \nabla \times A - A \cdot \nabla \times B \quad (2-53) \]

\[ A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B) \quad (2-54) \]

For the same reason mentioned in derivation of Eq. (2-31), the surface integral in Eq. (2-52) is set to zero to satisfy the weak boundary condition of \( H \) in the tangential direction, so \( G_i \) becomes:

\[ G_i = \int_V \nu \nabla \times A \cdot \nabla \times N_i^{ed} - N_i \cdot J_\phi + N_i^{ed} \cdot \left( \sigma \left( \frac{\partial A}{\partial t} + \nabla \phi \right) \right) dV = 0. \quad (2-55) \]

We also have

\[ \nabla \times A = \nabla \times \left( \sum_{j=1}^{12} N_j^{ed} A_j \right) = \sum_{j=1}^{12} (\nabla \times N_j^{ed}) A_j, \quad (2-56) \]
\[
\n\nabla \phi = \nabla \left( \sum_{j=1}^{N} N_j \phi_j \right) = \sum_{j=1}^{N} (\nabla N_j) \phi_j. \tag{2-57}
\]

Substitute Eqs. (2-56), and (2-57) to Eq. (2-50), \( G_i \) becomes:

\[
G_i = \sum_{i=1}^{N_e} \sum_{j=1}^{12} \left( \nu \nabla \times N_i^e \cdot \nabla \times N_j^e A_j + \sigma N_i^e \cdot N_j^e \frac{\partial A_j}{\partial t} \right) dV
+ \sum_{i=1}^{N_p} \sum_{j=1}^{8} \left( \nabla N_i^e \cdot \nabla N_j^e \phi \right) dV - \sum_{i=1}^{N_e} \int_{V_i} N_i^e \cdot F_{o} dV = 0 \tag{2-58}
\]

Eq. (2-27) is changed to:

\[
G_{d_i} = -\int_{V} \nabla \nabla \cdot \left( \sigma \left( \frac{\partial A}{\partial t} + \nabla \phi \right) \right) dV + \int_{S} \left( \sigma \left( \frac{\partial A}{\partial t} + \nabla \phi \right) \right) \cdot n dS. \tag{2-59}
\]

In the proceeding above, the following vector operation and the Gauss theorem are used.

\[
\nabla \cdot (f A) = \nabla f \cdot A + f \nabla \cdot A \tag{2-60}
\]

The second right term in Eq. (2-55) is set to zero to satisfy the electric conservation law weakly. So \( G_{d_i} \) becomes:

\[
G_{d_i} = -\sum_{i=1}^{N_e} \sum_{j=1}^{12} \int_{V_i} \nabla \nabla \cdot N_i^e \cdot N_j^e \frac{\partial A_j}{\partial t} dV - \sum_{i=1}^{N_p} \sum_{j=1}^{8} \int_{V_i} \nabla \nabla \cdot N_i^e dV \phi_j = 0 \tag{2-61}
\]

The integrations are evaluated numerically as described in the previous section.

By writing equations (2-58) and (2-61) in matrix form we have:

\[
KA + S \frac{\partial A}{\partial t} + P^T \phi = b \tag{2-62}
\]

\[
P \frac{\partial A}{\partial t} + Q \phi = 0 \tag{2-63}
\]
Where $P^T$ is the transpose of $P$. Using backward Euler method with constant time step size $\Delta t$, the time derivatives are discretized as follows:

$$\left( \frac{\partial A}{\partial t} \right)^T = \frac{A^t - A^{t-\Delta t}}{\Delta t}$$  \hspace{1cm} (2-64)

Substituting Eq. (2-64) into Eqs. (2-62) and (2-63) results in the following equations:

$$(K + S^+)A^t + P^T \varphi^t = b + S^+ A^{t-\Delta t}, \hspace{1cm} (2-65)$$

$$PA + Q^+ \varphi^t = PA^{t-\Delta t}, \hspace{1cm} (2-66)$$

$$S^+ = \frac{S}{\Delta t}, \hspace{1cm} Q^+ = \Delta t Q,$$

which can be combined in a single matrix equation as shown graphically in Fig. 2-7-a. The coefficient matrix is symmetric and positive definite. In the linear case one matrix solve per step gives the solution. In the nonlinear case, only $H$ depends on $A$, and similar to the previous section, the equation can be solved using the Newton-Raphson iteration as explained in the previous section. The graphic illustration of the linearized equation is shown in Fig. 2-7-b and Fig. 2-7-c shows the algorithm of solving the nonlinear equation within period $T$ of time.
Fig. 2.7. (a) Matrix equations for linear case, (b) nonlinear case, (c) solution algorithm for nonlinear case. $T$ is the time period of analysis.

Form Eq. (2-46) we have $K' = \frac{\partial K}{\partial A}$, so

\[
K'_{ij} = \frac{\partial K_{ij}}{\partial A_j} = \frac{\partial K_{ij}}{\partial V} \frac{\partial B}{\partial A_j} A_j.
\]  
(2-67)

From Eq. (2-58) we have

\[
K_{ij} = \sum_{i=1}^{N_{el}} \sum_{j=1}^{12} \int \left( \nabla \times N_{i,ed} \cdot \nabla \times N_{j,ed} \right) dV.
\]  
(2-68)

\[
\frac{\partial K_{ij}}{\partial V} A_j = \sum_{i=1}^{N_{el}} \sum_{j=1}^{12} \int \left( \nabla \times N_{i,ed} \cdot \nabla \times N_{j,ed} \right) dV A_j = 
\sum_{i=1}^{N_{el}} \int \left( \nabla \times N_{i,ed} \cdot \nabla \times N_{j,ed} \right) dV = 
\sum_{i=1}^{N_{el}} \int \left( \nabla \times N_{i,ed} \cdot B \right) dV
\]  
(2-69)

or

\[
\frac{\partial K_{ij}}{\partial V} A_j = \sum_{i=1}^{N_{el}} \int \left( \nabla \times N_{i,ed} \cdot B \right) dV
\]  
(2-70)

\[
\frac{\partial B}{\partial A_j} = \frac{\partial B}{\partial A_j} \frac{\partial B}{\partial A_j} = - \frac{1}{2B} \frac{\partial \left( B \cdot B \right)}{\partial A_j} = - \frac{1}{2B} 2B \cdot \frac{\partial B}{\partial A_j} = - \frac{B}{B} \cdot \frac{\partial \left( B \cdot B \right)}{\partial A_j} = - \frac{1}{B} B \cdot \nabla \times N_{j,ed}
\]  
(2-71)

\[
\frac{\partial V}{\partial B} = \frac{\partial H}{\partial B} = - \frac{1}{B^2} \left( \frac{dH}{dB} - B - H \right) = \frac{1}{B} \left( V - \frac{dH}{dB} \right).
\]  
(2-72)

and finally

\[
K'_{ij} = \sum_{i=1}^{N_{el}} \frac{1}{B^2} \left[ \frac{dH}{dB} - V \right] \sum_{j=1}^{12} \int \left( B \cdot \nabla \times N_{j,ed} \right) dV.
\]  
(2-73)
2.5.3 Two-Dimensional Discretization Using Triangular Elements

In this thesis, only two-dimensional magnetic field analysis with first order triangular elements was applied to the problem, thus we limit the discussion on to discretization with this type of elements. The domain of analysis is first discretized into a triangular mesh such as the one seen in Fig. 2.8. Similar to the three-dimensional case. In 2-D Cartesian coordinate system, the current source and the vector potential have only one component, the \( z \) component, while the magnetic flux density lies on the 2-D plane. This simplifies the fundamental equations as follows.

![Fig. 2.8. Triangular elements in 2-D discretization. The unknowns are the field values at the nodes.](image)

In first-order triangulate elements, referring to Fig. 2.9, the shape functions in the global coordinates are defined as follows:
\[ N_i = \frac{S_i}{S}, \quad i = 1,2,3. \]
\[ S = S_1 + S_2 + S_3. \]

The area of a triangle with is given by the following equation [4]:
\[ S = \frac{1}{2} \det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \]
where \((x_i,y_i)\) are the coordinates of the triangle vertices. In the local coordinate system, the shape functions take the following form:
\[ N_i = \begin{cases} \xi & i = 1 \\ \eta & i = 2 \\ 1 - \xi - \eta & i = 3 \end{cases} \]

The Gauss integration method discussed earlier for the hexahedral elements in local coordinates is not applicable to integration over triangle. Integration over elements has its own rule described in TABLE 2.3 [66].

**TABLE 2.3. Integration points and weights for integration over the reference element**

<table>
<thead>
<tr>
<th>Number of points, (n)</th>
<th>Points, ((\xi_k, \eta_k))</th>
<th>Weight, (w_k)</th>
<th>exact up to order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((1/3,1/3))</td>
<td>1/2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>((1/2,0), (0,1/2),(1/2,1/2))</td>
<td>1/6,1/6,1/6</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>((1/5,1/5), (1/5,3/5) (3/5,1/5),(1/3,1/3))</td>
<td>25/96, 25/96, 25/96, -27/96</td>
<td>3</td>
</tr>
</tbody>
</table>
Recall the fundamental coupled circuit-field equation, Eqs. (2-23) and (2-24). If the magnetic core is laminated and the coils are stranded, it is justified to neglect the eddy current term:

\[ \nabla \times (\nu \nabla \times A + M) = \frac{NI}{S} \hat{I}, \quad (2-74) \]

\[ L \frac{dI}{dt} + RI - \frac{N}{S_c} \int_V \frac{\partial A}{\partial t} \cdot \hat{I} \, dv = V, \quad (2-75) \]

where \( M \) is the magnetization of the permanent magnet and it appears in the Ampere law from the definition of \( H \), and \( S_c \) is the coil cross section. In 2-D, \( \hat{I} \) is constant and it is the unit vector along \( z \)-axis if the plane of analysis is \( x-y \) plane. We expect the equations to become simpler in 2-D and in fact they do but not in a straightforward way. The simplification is explained in the following proceedings:

\[ A = (0,0,A) \Rightarrow \nabla \times A = \left( \frac{\partial A}{\partial y}, - \frac{\partial A}{\partial x}, 0 \right) \Rightarrow \nabla \times (\nu \nabla \times A) = (0,0,- \frac{\partial A}{\partial x} \left( \frac{\partial A}{\partial x} \right) - \frac{\partial A}{\partial y} \left( \frac{\partial A}{\partial y} \right)) \]

\[ M = (M_x,M_y,0) \Rightarrow \nabla \times M = (0,0,- \frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y}). \]

Therefore the coupled equations become

\[ \frac{\partial A}{\partial x} \left( \frac{\partial A}{\partial y} \right) + \frac{\partial A}{\partial y} \left( \frac{\partial A}{\partial x} \right) + \frac{NI}{S} = \left( \frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y} \right), \quad (2-76) \]

\[ L \frac{dI}{dt} + RI - \frac{N h}{S_c} \int_{S_c} \frac{\partial A}{\partial t} \, dS = V. \quad (2-77) \]

Applying the weighted residual Galerkin method to Eqs. (2-76) and (2-77)

\[ \int_S N_i \left( \frac{\partial}{\partial x} \left( \frac{\partial A}{\partial y} \right) + \frac{\partial}{\partial y} \left( \frac{\partial A}{\partial x} \right) + \frac{NI}{S} \right) \, dS = \int_S N_i \left( \frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y} \right) \, dS, \quad (2-78) \]
where $S$ is the entire domain of analysis. By defining $\hat{\nu} = \begin{bmatrix} \nu_{yy} & 0 \\ 0 & \nu_{ss} \end{bmatrix}$, the equation can be written in a more compact form as follows:

$$
\int_{S} N_i \left( \nabla \cdot (\hat{\nu} \nabla A) + \frac{NI}{S} \right) dS = \int_{S} N_i \left( \frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y} \right) dS,
$$

(2-79)

The left-hand-side can be simplified as follows:

$$
\int_{S} N_i \nabla \cdot (\hat{\nu} \nabla A) dS = \int_{S} \nabla \cdot (N_i \hat{\nu} \nabla A) dS - \int_{S} \nabla N_i \cdot (\nabla A) dS
$$

(2-80)

In the proceeding above, the following vector identity was used:

$$
\nabla \cdot (\rho B) = \nabla \cdot \rho B + \rho \nabla \cdot B
$$

For the same reason mentioned in the derivation of 3-D equations, the first integral becomes zero. Therefore

$$
-\int_{S} \nabla N_i \cdot (\nabla A) dS + \frac{N}{S_c} \int_{S_c} N_i I dS = \int_{S} N_i \left( \frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y} \right) dS,
$$

(2-81)

or in a compact form:

$$
\sum_{i \in S} \int_{S} \nabla N_i \cdot (\nabla A) dxdy + Q_i I = F_i,
$$

(2-82)

where $Q_i$ and $F_i$ can be easily identified by comparing Eq. (2-82) to Eq. (2-81). In first order triangular elements we have

$$
A = \sum_{j=1}^{3} A_j,
$$

so the fully discretized equations becomes

$$
\sum K_{ij} A_j' + Q_i I' = F_i,
$$

(2-83)
\begin{equation}
L(I' - I'^{-\Delta t}) + R\Delta t I - \frac{Nh}{S_c} (A' - A'^{-\Delta t}) d\gamma = \Delta t V. \tag{2-84}
\end{equation}

In Eq. (2-83), \( K_{ij} \) is given by the following equation:

\begin{equation}
K_{ij} = \sum_{ie} \sum_{j=1}^{3} \nabla N_j \cdot (\nu_k \nabla N_j). \tag{2-85}
\end{equation}

The matrix form of Eqs. (2-83) and (2-84) can be seen in Fig. 2-10.

Fig. 2.10. Discretized field-circuit equations in matrix form for the linear case.

In the nonlinear case, similar to the 3-D derivations in the previous section, the matrix equation for the Newton-Raphson method is obtained. The equation is shown in Fig. 2-11.

\begin{equation}
H^* A = H_k^* A_k + \frac{\partial (H^* A)}{\partial A} dA = H_k^* A_k + \left( H_k^* + \frac{\partial H^*}{\partial A} \right) dA \tag{2-86}
\end{equation}

It can be shown that

\begin{equation}
K'_{ij} = \sum_{ie} \sum_{j=1}^{3} \{\nabla N_i \cdot B\} \{\nabla N_j \cdot (\nu_k B)\}. \tag{2-87}
\end{equation}
The right hand side is known thus the system of equations to be solved.

\[ Ax = b \]  \hspace{1cm} (2-88)

Once the system is solved and the vector potential is found for all the unknown nodes. The magnetic flux density within each element can be simply computed according to the following equation:

\[ B = \nabla \times A \]  \hspace{1cm} (2-89)

where \( A = (0,0, \sum_{j=1}^{3} A_j) \).
2.6 Iterative Sparse Matrix Solvers for Finite Element Equations

The coefficient matrix is usually too large to reside in the computer memory as a regular matrix, but since it is sparse, i.e. most of the entries are zero, only the nonzero entries are stored. These way matrix-vector products are efficiently performed as the operation is done only on the nonzero entries [45]. The matrix is also positive symmetric and positive definite. Iterative methods are used to solve such matrices and methods based on the conjugate gradient method are the best. Although there exists other methods such as Jacobi and Gauss-Seidel which are also iterative, the time of conjugate gradient methods are much faster. In the following section, several methods which are used in finite element computation are explained.

2.6.1 Conjugate Gradient (CG) Method

The conjugate gradient method solves Eq. (2-88) by minimizing the following functional [15], [45]:

\[ \phi = \frac{1}{2} x^T A x - x^T b \]  

(2-90)

The method starts from initial guess \( x_0 \) and creates the Krylov subspace as follows:

\[ K^n(A, x_0) = \text{span} \{ x_0, A x_0, A^2 x_0, \ldots, A^{n-1} x_0 \}, \]

so, at step \( n \) the solution is searched in the Krylov subspace. This is achieved by finding a set of conjugate directions in this subspace. The conjugacy condition is
\[ p_i^T A p_j = 0 \quad i \neq j. \] (2-91)

The solution \( x \) is approximated by a linear combination of these conjugate directions:

\[ x = \sum_{i=0}^{n-1} \alpha_i p_i \] (2-92)

If \( p_i \) were known, \( \alpha_i \) could be easily calculated by substituting \( x \) into Eq. (2-90).

\[ \sum_{i=0}^{n-1} \alpha_i A p_i = b \]

\[ \sum_{i=0}^{n-1} \alpha_i p_i^T A p_i = p_i^T b \]

\[ \alpha_k = \frac{p_k^T b}{p_k^T A p_k} \] (2-93)

By choosing the search directions \( p_k \) such that they are conjugates with respect to \( A \),

\[ r_k = b - A x_k \] (2-94)

\[ \nabla \phi = A x - b = -r \] (2-95)

At each step, instead of searching in the direction of \( r_k \), a computed is made out of \( r_k \) such that it is orthogonal to all previous searches.

\[ p_i^T A p_j = 0 \quad i \neq j \]

\[ p_k = r_k - \sum_{i<k} p_i^T A r_k p_i \] (2-96)

\[ x_{k+1} = x_k + \alpha_k p_k \] (2-97)

Now \( \alpha_k \) is given by Eq. (2-93). It can be shown that in the conjugate gradient method, the residuals of all steps are orthogonal and this guarantees that if the matrix has size \( N \), after at most \( N \) iterations, the exact solution is found because in
N-dimensional space, maximum $N$ orthogonal vectors orthogonal can be found and this means the next residual would be zero and this means the exact solution has been reached. In practice with much less number of iterations a good approximate of the solution is found.

The resulting algorithm.

\[ r_0 := b - Ax_0 \]
\[ p_0 := r_0 \]
\[ k := 0 \]
repeat
\[ \alpha_k := \frac{r_k^T r_k}{p_k^T A p_k} \]
\[ x_{k+1} := x_k + \alpha_k p_k \]
\[ r_{k+1} := r_k - \alpha_k A p_k \]
If \( r_{k+1} \) is smaller than a desired criterion exit loop.
\[ \beta_k := \frac{r_{k+1}^T r_{k+1}}{r_k^T r_k} \]
\[ p_{k+1} := r_{k+1} + \beta_k p_k \]
\[ k := k + 1 \]
End repeat.

Although the method converges theoretically and convergence is guaranteed in no more than $N$ iteration, in practice for large matrices due to numerical error the convergence may not be achieved at all. This is determined by the condition number of a matrix which is defined as the ration of the largest to the smallest eigenvalues.
The closer to unity the condition number, the faster convergence. There is an improvement to the conjugate gradient method which help achieve a faster convergence by solving a system whose matrix has a lower condition number. The method is called preconditioned conjugate gradient method the specific precondition using the incomplete Cholesky factorization, the method is known as the incomplete Cholesky conjugate gradient (ICCG) method. The method is explained in the following section.

2.6.2 Incomplete Cholesky Conjugate Gradient (ICCG) Method

The symmetric positive definite matrix $A$ can be factorized into the following form [45]:

$$A = LDL^T$$  \hspace{1cm} (2-98)

The algorithm is given as follows:

$$D_j = A_{jj} - \sum_{k=1}^{j-1} L_{jk}^2 D_k$$  \hspace{1cm} (2-99)

$$L_{jj} = \frac{1}{D_j} \left( A_{jj} - \sum_{k=1}^{j-1} L_{kj} L_{jk} D_k \right)$$  \hspace{1cm} (2-100)

Where $L$ is a lower triangular with all diagonal elements being one, and $D$ is a diagonal matrix. Since the matrix is positive definite all diagonal elements of $D$ are positive. Thus $A$ can also be written as follows:

$$A = (L\sqrt{D})(L\sqrt{D})^T = MM^T$$  \hspace{1cm} (2-101)

$$MM^Tx = b$$  \hspace{1cm} (2-102)
Since \( M \) is triangular, Eq. (102) can be solved with low cost in two steps using forward elimination and back substitution respectively:

\[
My = b \tag{2-103}
\]

\[
M^{T}x = y \tag{2-104}
\]

The difficulty is that computing the complete Cholesky factorization of large matrices, even sparse, is not practical as it may result in too many fill-ins. One popular solution is to factor the matrix incompletely, i.e. if \( M \) has the same non-zero pattern as \( A \) by just setting the corresponding entries of \( M \) to zero without computing them. Now we have

\[
A = MM^{T} \tag{2-105}
\]

The algorithm of factorization requires a minor change to ensure that the resulting matrix is still positive definite. This is done by the so-called shifting factor \( \gamma \geq 1 \). In practice \( 1.2 \geq \gamma \geq 1.05 \) is a good choice. Large values reduce the effect of preconditioning.

\[
D_{j} = \gamma A_{jj} - \sum_{k=1}^{j-1} L_{jk}^{2} D_{k}, \tag{2-106}
\]

because the equality may not be held anymore. However this approximation is not going to affect the solution of the original system because the matrix is only used as preconditioned as follows:

The original equation is pre-multiplied by the inverse of \( M \):

\[
M^{-1}Ax = M^{-1}b \tag{2-107}
\]

Note that there is no need to compute the inverse of the matrix. In order to preserve the symmetry,
We define \( y = Mx \), thus

\[
M^{-1}AM^{-T}y = M^{-1}b
\]  
(2-108)

It can be shown that \( M^{-1}AM^{-T} \) has a lower condition number than \( A \). In fact if the factorization is complete, the matrix becomes identity matrix and the system is solved. Since the factorization is incomplete, the CG method is applied to Eq. (2-108), bearing in the mind that for computing \( x = M^{-1}y \), in practice system \( Mx = y \) is solved and this is cheap as \( M \) is triangular.

The resulting algorithm of ICCG method then becomes [45]

\[
\begin{align*}
    r_0 & := b - Ax_0 \\
    z_0 & := M^{-1}r_0 \\
    p_0 & := z_0 \\
    k & := 0 \\
    \text{repeat} \\
    \alpha_k & := \frac{r_k^T z_k}{p_k^T Ap_k} \\
    x_{k+1} & := x_k + \alpha_k p_k \\
    r_{k+1} & := r_k - \alpha_k Ap_k \\
    \text{If } r_{k+1} \text{ is smaller than a desired criterion exit loop} \\
    z_{k+1} & := M^{-1}z_k \\
    \beta_k & := \frac{z_{k+1}^T r_{k+1}}{z_k^T r_k} \\
    p_{k+1} & := \frac{r_{k+1}^T z_{k+1}}{r_k^T r_k} \\
    p_{k+1} & := r_{k+1} - \beta_k p_k \\
    k & := k + 1 \\
    \text{End repeat.}
\end{align*}
\]
2.6.3 Complex Orthogonal ICCG (COICCG) Method

The CG and ICCG methods are applicable to complex Hermitian positive matrix as well. For such matrices the Cholesky factorization takes the following form:

\[ A = M M^H, \quad (2-109) \]

where H denotes the Hermitian transpose or conjugate transpose. Accordingly the product in CG and ICCG algorithms become Hermitian product as the vectors and also the matrix are complex. This guarantees that products such as \( r_k^H r_k \) or \( p_k^H A p_k \) become real numbers.

In the finite element method, when the system of equations is chosen to solved in the frequency domain, the system of equation may become complex symmetric. A necessary condition for the existence of Cholesky decomposition for a complex matrix is that the matrix has to be Hermitian. Furthermore the CG method does not converge if the matrix is not Hermitian. Although the problem is complicated, mathematicians have found an easy solution to the problem though understanding the trick is not straight forward [16]. The method is called which is called Complex Orthogonal ICCG (COICCG). The algorithm is very similar to the CG and ICCG algorithms only that the products such as \( r_k^T r_k \) or \( p_k^T A p_k \) may become complex numbers. In addition, for complex symmetric matrix A, the preconditioner \( M \) satisfies the following equation:

\[ A \approx M M^T \]
2.7 Magnetic Nodal Force Calculation

In motors, it is often required to calculate the torque. In addition in magneto-mechanical problems, definition and knowledge of nodal force calculation is essential. These are methods to calculate the total force over a body but for magneto-mechanical analysis, nodal force is required. One popular method of magnetic nodal force calculation is the use of Chu-type magnetic stress tensor. The stress tensor is given by the following equation [27]:

\[
T_{ij}^m = B_i H_j - \delta_{ij} \left( \mathbf{B} \cdot \mathbf{H} - \int_0^B h_0 \cdot db \right),
\]

where \(T\) is the Chu-type Maxwell stress tensor, \(B\) is the flux density, \(H\) is the magnetic field strength, and \(\delta\) is the Kronecker’s delta. The volume force density then becomes the divergence of the tensor:

\[
f^\Omega = \nabla \cdot T^m
\]

In addition to the volume force density, there exists surface force density or surface traction force. This force is exerted on the material interface, iron-air interface for example.

\[
f^\Gamma = (T^m_{2} - T^m_{1}) \cdot \mathbf{n} \quad on \quad \Gamma
\]

Nodal force can be calculated by multiplying the volume force density by the nodal shape function \(N_n\) and the integration over the entire domain.

\[
F^m_n = \sum \int_{V_n} N_n \nabla \cdot T^m \, dv
\]

Using divergence theorem and combining both volume and surface forces, Eq. (2-74) is simplified to the following form [27]:

\[
\]
\[
F^m_n = -\sum_{\epsilon} \int_{V_{\epsilon}} T^m_n \nabla N_n \, dv
\]

(2-114)

However since the finite element shape functions associated with each node is only nonzero in elements that share the nodes, in practices the integration is only done over those elements.

### 2.8 Magnetostriction

Magnetostriction is a property of ferromagnetic materials that is observed as a change in shape or dimensions during the process of magnetization. The phenomenon was first identified in 1842 by James Joule when observing a sample of iron [2]. In principle the change of dimension (deformation) occurs as a result of orientation and expansion of magnetic domains. Since in macroscopic Maxwell equations the details in domain scale is not considered, the computation of the deformation is also impossible. The effect however, can be measured experimentally as a function of magnetic flux density. Magnetostriction is a three-dimensional phenomenon, i.e. the change of dimension happens in three dimensions even though the flux is applied in one dimension. In fact experiments reveals that the change of dimensions is, usually, such that the volume remain unchanged. For this reason, magnetostriction which such assumption is called *isochoric* magnetostriction [23]. The measurement is usually made in one dimension, i.e., in the dimension of the applied field. The ration of the increase of the dimension on the original dimension which has no dimension is often noted as \( \dot{\lambda} = \dot{\lambda}(B) \) [26].
When $B$ is parallel with $x$ direction, the strain components are:

$$\lambda_x = \lambda$$

If the magnetostriction is isotropic, we have $\lambda_y = \lambda_z$. To have the volume unchanged, the condition is:

$$l_x l_y l_z = l_x (1 + \lambda) l_y (1 + \lambda_y)^2 .$$

Since the value of $\lambda$ is very small, the following relation is obtained.

$$\lambda_y = \lambda_z = \frac{-\lambda}{2} .$$
2.9 Magnetostriction Strain and Stress Tensors

Assuming isotropic and isochoric magnetostriction [23], when \( B \) is along \( x \) axis, the MS strain tensor \( \varepsilon^{ms} \) in Voigt notation is given by the following equation:

\[
\varepsilon^{ms} = \begin{bmatrix}
\lambda & -\frac{\lambda}{2} & -\frac{\lambda}{2} & 0 & 0 & 0
\end{bmatrix}^T,
\]

(2-115)

where \( \lambda = \lambda(B, \sigma_B) \) is the MS strain in the direction of the flux density \( B \) under mechanical stress \( \sigma_B \) (positive sign for tensile stress) applied in \( B \) direction, measured or modeled as in [28] and [26], respectively. For an arbitrary direction of \( B \), using tensor transformation rule, the following expression is obtained:

\[
\varepsilon^{ms} = \lambda S,
\]

(2-116)

\[
S_{ij} = \frac{1}{2B^2} \left( 3B_i B_j - \delta_{ij} B^2 \right),
\]

(2-117)

where \( S \) is called the MS direction tensor (it depends only on the direction of \( B \)), \( \delta_{ij} \) is the Kronecker delta, and \( \varepsilon^{ms} \) is in matrix form. (Throughout this paper, when a symmetric tensor appears in bold italic letter, it is in Voigt notation whereas italic font is used for the matrix form).

The system may have both MS and elastic strains. This can be treated as if the body is first free to reach its new dimensions by MS strain and then other forces come into action and cause the elastic deformation \( \varepsilon^{\text{elast}} \) to the body that already has the inelastic strain \( \varepsilon^{ms} \). It is obvious that \( \varepsilon = \varepsilon^{ms} + \varepsilon^{\text{elast}} \), where \( \varepsilon \) is the total strain. Only the elastic strain contributes to mechanical stress [26]. According to Hook’s law
\[ \sigma = D\varepsilon^{\text{elas}} = D\varepsilon - D\varepsilon^{\text{ms}}, \quad (2-118) \]

where \( \sigma \) is the stress tensor and \( D \) is the 6×6 stiffness tensor widely used in structural analysis. Now we define the following tensor [33]:

\[ T^{\text{ms}} = -D\varepsilon^{\text{ms}}. \quad (2-119) \]

Therefore the stress tensor becomes

\[ \sigma = D\varepsilon + T^{\text{ms}}. \quad (2-120) \]

We refer to \( T^{\text{ms}} \) as the magnetostriction stress tensor which is the internal stress due to MS. After performing the matrix-vector multiplication in (2-119) we have [33]

\[ T^{\text{ms}} = \frac{-E\lambda}{(1+\nu_{\rho})}S, \quad (2-121) \]

where \( E \) and \( \nu_{\rho} \) are Young’s modulus and Poisson’s ratio respectively.

### 2.10 Magnetostriction Anisotropy

In the above derivations we assumed that magnetostriction is isotropic. However, measurements on electrical steel sheets reveals string anisotropy even in magnetically isotropic materials (Fig. 2.14). The anisotropy is explained by the effect of tensile stress during the rolling process of the sheets. The tensile stress usually reduces magnetostriction strain thus, magnetostriction strain in the rolling direction becomes less than in the transverse direction.
2.11 Magnetostriction Dependence on Mechanical Stress

Magnetostriction shows strong dependence on the mechanical stress. The stress could be applied externally or could be induced by an applied magnetic field. Some measurements results are shown below. It is important to consider this dependence in the analysis, but if the stress is low, let’s say lower than 1.0 MPa, neglecting its effect could be justified.

The measurements are often made in one direction, i.e., the stress is applied in the same direction the field. However, since the magnetostriction in other directions is also related to the longitudinal direction, these curves give some information about the effect of stress on the magnetostriction in the other two dimensions as well.
2.12 Magnetostriction Nodal Force Calculation

The measured magnetostriction parameter is under uniform field and without any constraint. In typical electrical devices, the field inside the ferromagnetic core is usually non-uniform. In addition, the core may be under constraints, for example fixed at some points. In this case, in order to known the deformation of the structure due to magnetostriction, a structural analysis using the finite element method is required which demands for the knowledge of nodal forces, i.e., forces applied at points whose displacements are to be known. We mentioned earlier that the magnetic force can be calculated from the magnetic stress tensor.
Calculating deformations of free elements using $\lambda$ and dimensions

Calculating equivalent force from deformations and element stiffness matrix.

Summing up the forces

Calculating equivalent force using the proposed method.

Summing up the forces

Fig. 2.16. Comparison between magnetostriction force calculation methods, (a) conventional method, (b) proposed method, (c) comparison of numerical results.

In Chapter 3 we will show that the magnetostriction force can be calculated from the magnetostriction tensor exactly the same way the magnetic force is calculated from the magnetic stress tensor. Thus

$$ F_n^{ms} = -\sum_v \int_T \nabla e_{nv}^m N_n d\nu, \quad (2-122) $$

$$ S_{ij} = \frac{1}{2B^2} \left( 3B_i B_j - \delta_{ij} B^2 \right), \quad (2-123) $$

where $F_n^{ms}$ is the magnetostriction force at node $n$ and $N_n$ is the shape function of the node [33]. Fig. 2.16 compares the proposed method with the conventional method described in [26].
### 2.13 Core Loss Calculation

The core loss characteristics of electrical steel of a particular grade can be measured by applying a uniform alternating sinusoidal flux density at a given frequency [62]. Such a measured data is shown in Fig. 2.17-(a).

![Fig. 2.17. (a) Core loss characteristics of electrical steel of grade 50A400 v.s. peak value of alternating sinusoidal flux density at various frequencies, (b) core loss v.s. stress at 1.0 T, 400 Hz.](image)

Bertotti model of separation of core losses is widely used for the core loss estimation [62], [65]. In this model, all measured data in Fig. 2.17-(a) can be approximated with a function of peak value of flux density $B$ and the frequency $f$ with four parameters as follows:

$$\frac{w(B, f)}{f} = k_h B^\alpha + k_a B^{1.5} \sqrt{f} + k_e B^2 f$$  \hspace{1cm} (2-124)

where $k_h$, $k_a$, and $k_e$, are constants associated with the hysteresis, excess, and eddy current loss, respectively, and $\alpha$ is the exponent constant of the hysteresis part. The
constants, which are often considered independent of \( B \) and \( f \), can be calculated from experimental data using curve fitting techniques [62]. The core loss also depends on the compressive stress as can be seen in Fig. 2.17-(b) [63]. To accommodate the effect of stress on the loss into Bertotti’s method, we assume that only the hysteresis loss depends on stress. The modified expression of the core loss then takes the following form:

\[
\frac{w(B, f, \sigma)}{f} = k_n (1 + C_\sigma) B^n + k_p B^{1.5} \sqrt{f} + k_s B^2 f,
\]

(2-125)

where \( C_\sigma \), assumed independent of \( B \) and \( f \), is stress-dependency coefficient and depends on \( \sigma_{eq} \), and equivalent scalar stress, given by the following equation:

\[
\sigma_{eq} = \sum_{i=1}^{3} \sum_{j=1}^{3} S_{ij} \sigma_{eq}.
\]

(2-126)

where \( S \) is the magnetostriction direction tensor discussed in Section 2.9. The derivation and meaning of (2-126) will be presented in Section 4.3. Note that for hydrostatic pressure, \( \sigma_{eq} = 0 \) according to (2-126).

Equation (2-15) can be rewritten in the following form:

\[
w(B, f, \sigma) = w(B, f, 0) + k_n C_\sigma B^n f.
\]

(2-127)

The parameter \( C_\sigma \) is defined by three parameters \( k_1, k_2, \sigma_1 \), as shown in Fig. 2.18.

![Fig. 2.18. Piecewise linear approximation of stress-dependency coefficient.](image-url)
The flux density is, in general, neither purely sinusoidal nor purely alternating but it contains both time and spatial harmonics. Fig. 2.19-(a) shows a typical locus of flux density at one of the tooth tip of a motor stator. Within the teeth and in the back iron, the flux is nearly alternative and has smaller time harmonics. For an elliptical locus, we use one of a conventional method and calculate the losses for minor and major axes of ellipse separately and sum up the results [64]. To this end, the actual locus is decomposed to its Fourier components where each component forms an elliptical locus (see Fig. 2.19-(b)). We also assume that the minor hysteresis loops has the same shape as the major loop and this allows us to calculate the loss due to harmonics by plugging the harmonic amplitudes into (2-124). This assumption might be disputable because the in practice the minor loops depend on the peak value of the main harmonic [64]. There are methods taking account of the effect the flux peak value on the minor loops [64] but needs extra experimental data which we lack.

![Fig. 2.19](image)

Fig. 2.19 (a) a general locus of flux density with time and space harmonics, (b) the locus of 1\textsuperscript{st}, 3\textsuperscript{rd} and 5\textsuperscript{th} harmonics of the same flux density.
Chapter 3 Static Structural Analysis

3.1 Introduction

Consider the simple structure below. An elastic bar discretized into rectangles, fixed on one end and under a set of nodal force. We are interested in the displacement of each node due to the given force.

If the elasticity is linear, we can write a relation between the force and the displacements in a general form as follows:

\[
\begin{bmatrix}
    u_{i,x} \\
    u_{i,y} \\
    u_{i,z}
\end{bmatrix} = \sum_{j=1}^{N_{\text{node}}} \begin{bmatrix}
    F_{i,x} \\
    F_{i,y} \\
    F_{i,z}
\end{bmatrix} A_{ij}
\]

If all unknowns are ensemble in vector \( \mathbf{u} \), and all the force components in vector \( \mathbf{f} \), we have

\[ \mathbf{u} = A\mathbf{f} \quad (3-1) \]

If \( A \) is known, \( \mathbf{u} \) can be easily computed. However in the finite element not \( A \) but its inverse can be computed. Most references introduce the discretized form of the static equation, without the derivation, as follows [46]:

\[ Ku = f \quad (3-2) \]

where \( K \) are global mass, damping and stiffness matrices, respectively, \( \mathbf{u} \) is the displacement, and \( \mathbf{f} \) is the loading. The global stiffness matrix is constructed from element stiffness matrix. The equations for element stiffness matrix \( K_{ij} \) is given as follows:
\[ K_{ij} = \int_{V_i} B_i^T D B_j dv, \quad (3-3) \]

\[
B_j = \begin{bmatrix}
\frac{\partial N_j}{\partial x} & 0 & 0 \\
0 & \frac{\partial N_j}{\partial y} & 0 \\
0 & 0 & \frac{\partial N_j}{\partial z} \\
\frac{\partial N_j}{\partial y} & \frac{\partial N_j}{\partial x} & 0 \\
0 & \frac{\partial N_j}{\partial z} & \frac{\partial N_j}{\partial x} \\
\frac{\partial N_j}{\partial z} & 0 & \frac{\partial N_j}{\partial x}
\end{bmatrix} \quad (3-4)
\]

where \( N_i \) and \( N_j \) are the nodal shape functions of nodes \( i \) and \( j \), respectively. In the Cartesian coordinate system, \( u \) and \( f \) have the following forms

\[
u = \begin{bmatrix}
u_x^{(1)} \\
u_y^{(1)} \\
u_z^{(1)} \\
u_x^{(2)} \\
u_y^{(2)} \\
u_z^{(2)} \\
\vdots
\end{bmatrix}, \quad f = \begin{bmatrix}F_x^{(1)} \\
F_y^{(1)} \\
F_z^{(1)} \\
F_x^{(2)} \\
F_y^{(2)} \\
F_z^{(2)} \\
\vdots
\end{bmatrix}, \quad (3-5)
\]

where subscripts \( x, y, \) and \( z \) denote the Cartesian components of the nodal force and displacements, and the superscripts \( (1) \) and \( (2) \) indicates the indices of unknown nodes. We would like to derive the equation from the continuum equation of equilibrium as it will reveal useful expressions for the calculation of magnetic, magnetostriction and thermal nodal forces. The following section describes the derivations in detail.
3.2 Equation of Static Equilibrium

The equations of equilibrium in the continuum media is [22]

\[ \nabla \cdot \sigma + f_v = 0, \]  
\[ \sigma (n_1 - n_2) = f_s, \]

where \( \sigma \) is the stress tensor, and \( f_v \) and \( f_s \) are volume and surface force densities respectively, and \( n_1 \) and \( n_2 \) are outward normal unit vectors to interface of material interface from regions 1 and 2 respectively [16].

\[ \sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}, \]

and

\[ \nabla \cdot \sigma = \begin{bmatrix} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} \\ \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} \\ \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \end{bmatrix} \]

The stress has to be expressed with respect to a reference level. Every elastic body may have an initial stress corresponding to an initial strain. Since we calculate
the stress with respect to the initial state of stress, Hooke’s law is, in the tensor notation, expressed as follows [33]:

\[ \sigma = C \cdot \varepsilon^{\text{ex}} = C : (\varepsilon - \varepsilon^{\text{ini}}) . \] (3-10)

Where \( C \) is the tensor of elasticity (of order four), and \( \varepsilon^{\text{ex}}, \varepsilon, \) and \( \varepsilon^{\text{ini}} \) are the excess, total and initial strain tensors, respectively. In the indicial notation, Hooke’s law can be written as follows:

\[ \sigma_{ij} = C_{ijkl} \varepsilon^{\text{ex}}_{kl} . \] (3-11)

The total strain in (3-10) is linked to the displacement \( u \) by the geometric law [16]:

\[ \varepsilon = \frac{1}{2} (\nabla u + (\nabla u)^T) . \] (3-12)

In the Cartesian coordinate system, we can write

\[
\varepsilon = \frac{1}{2} \begin{bmatrix}
\frac{\partial u_x}{\partial x} & \frac{\partial u_y}{\partial x} & \frac{\partial u_z}{\partial x} \\
\frac{\partial u_x}{\partial y} & \frac{\partial u_y}{\partial y} & \frac{\partial u_z}{\partial y} \\
\frac{\partial u_x}{\partial z} & \frac{\partial u_y}{\partial z} & \frac{\partial u_z}{\partial z}
\end{bmatrix}
+ \begin{bmatrix}
\frac{\partial u_x}{\partial x} & \frac{\partial u_y}{\partial x} & \frac{\partial u_z}{\partial x} \\
\frac{\partial u_x}{\partial y} & \frac{\partial u_y}{\partial y} & \frac{\partial u_z}{\partial y} \\
\frac{\partial u_x}{\partial z} & \frac{\partial u_y}{\partial z} & \frac{\partial u_z}{\partial z}
\end{bmatrix}
\] (3-13)

which becomes

\[
\varepsilon = \begin{bmatrix}
\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\
\varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\
\varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz}
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{2} \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) \\
\frac{1}{2} \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} + \frac{\partial u_z}{\partial z} \right) \\
\frac{1}{2} \left( \frac{\partial u_x}{\partial z} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial x} \right)
\end{bmatrix} . \] (3-14)
3.3 Finite Element Discretization of Equation of Static Equilibrium

In the finite element method, the displacement inside an element is interpolated with nodal shape function and follows:

\[ u = \sum_j N_j u_j , \quad (3-15) \]

where the sum extends over the vertex nodes of the element. The strain tensor in Voigt notation then becomes

\[
\dot{\varepsilon} = \sum_j \begin{bmatrix}
\frac{\partial N_j}{\partial x} u_{j,x} \\
\frac{\partial N_j}{\partial y} u_{j,y} \\
\frac{\partial N_j}{\partial z} u_{j,z}
\end{bmatrix}
= \sum_j B_j u_j , \quad B_j = \begin{bmatrix}
\frac{\partial N_j}{\partial x} & 0 & 0 \\
0 & \frac{\partial N_j}{\partial y} & 0 \\
0 & 0 & \frac{\partial N_j}{\partial z}
\end{bmatrix} .
\]

(3-16)

The arrow over the strain tensor indicates that the tensor is in the Voigt notation (vector form).

\[ \sigma = D\varepsilon^{\text{ex}} = D(\varepsilon - \varepsilon^{\text{ini}}) , \quad (3-17) \]

where \( D \) is the tensor of elasticity in the Voigt notation given by the following equation:
where $E$ is the Young’s modulus and $\nu$ is the Poisson’s ratio. The strain and stress tensors in the Voigt notation are as follows:

$$
D = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix}
1-\nu & \nu & 0 & 0 & 0 \\
1-\nu & \nu & 0 & 0 & 0 \\
1-\nu & 0 & 0 & 0 & 0 \\
(1-2\nu)/2 & 0 & 0 & 0 & 0 \\
(1-2\nu)/2 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
$$

(3-17)

If the initial strain is the thermal strain $\varepsilon^{\text{th}}$ and/or magnetostriction strain $\varepsilon^{\text{ms}}$, we have

$$
\varepsilon^{\text{init}} = \varepsilon^{\text{th}} + \varepsilon^{\text{ms}}.
$$

(3-19)

According to Eq. (3-10) implies that if a body is strained freely due to magnetostriction or due to thermal expansion, the state of stress of the body is zero. Substituting (3-19) in (3-10) and then in (3-6) results in the following equations:

$$
\nabla \cdot (C : \varepsilon) - \nabla \cdot (C : \varepsilon^{\text{th}}) - \nabla \cdot (C : \varepsilon^{\text{ms}}) + f_v = 0.
$$

(3-20)

Now we define the following tensor: [33]
\[ T^\text{th} = -C : \varepsilon^\text{th}, \quad (3-21) \]
\[ T^\text{ms} = -C : \varepsilon^\text{ms}, \quad (3-22) \]

where \( T^\text{th} \) and \( T^\text{ms} \) are called the thermal and magnetostriction stress tensors, respectively. The negative signs in (3-21) and (3-22) indicate that the tensors are not the initial strain tensors but sources of stress which caused the initial stress. Eq. (3-20) now becomes

\[ \nabla \cdot (C : \varepsilon) + \nabla \cdot T^\text{th} + \nabla \cdot T^\text{ms} + f_v = 0. \quad (3-23) \]

The volume force density could be the magnetic, gravitational, or some external volume force densities. Namely

\[ f_v = \rho g e_g + f^\text{m}_v, \quad (3-24) \]

where \( \rho g e_g \) and \( f^\text{m}_v \) are the gravitational, and magnetic force densities respectively. \( e_g \) is a unit vector pointing to the center of Earth.

Neglecting the dependence of the magnetic permeability on the stress, \( f^\text{m}_v \) is given by the following equation:

\[ f^\text{m}_v = \nabla \cdot T^\text{m} \quad (3-25) \]

\[ T^\text{m}_{ij} = B_i H_j - \delta^j_i \int_0^H b dH, \quad (3-26) \]

where \( T^\text{m} \) is the Maxwell stress tensor (also known as the magnetic stress tensor).

Eq. (3-23) can be discretized by nodal shape function as follows:

\[ \int_V N_i \nabla \cdot (C : \varepsilon) dV + \int_V N_i (\nabla \cdot T^\text{th} + \nabla \cdot T^\text{ms} + f_v) dV = 0. \quad (3-27) \]
From this point on, we limit the derivation to the 2-D case for simplicity. In 2-D plane strain we have

$$\varepsilon = G \begin{bmatrix} (1-v)\varepsilon_{xx} + v\varepsilon_{yy} & (1-2v)\varepsilon_{xy} \\ (1-2v)\varepsilon_{xy} & \varepsilon_{xx} + (1-v)\varepsilon_{yy} \end{bmatrix}$$

(3-28)

where

$$G = \frac{E}{(1+v)(1-2v)}$$

thus

$$N_i \nabla \cdot (C : \varepsilon) = G \left[ \begin{array}{c} \frac{\partial}{\partial x} \left[ N_i \varepsilon_{xx} + N_j \varepsilon_{xy} \right] + \frac{\partial}{\partial y} \left[ N_j \varepsilon_{xy} \right] \\ \frac{\partial}{\partial x} \left[ N_i \varepsilon_{xy} \right] + \frac{\partial}{\partial y} \left[ N_j \varepsilon_{yy} + (1-v)\varepsilon_{yo} \right] \end{array} \right]$$

$$= \nabla \cdot (N_i C : \varepsilon) - B_i^T D\varepsilon$$

where in 2-D,

$$B_i = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix}$$

(3-29)

Using vector identity and the divergence theorem,

$$\int_N \nabla \cdot (C : \varepsilon) dv = \int_N \nabla \cdot (N_i C : \varepsilon) dv - \int_B D\varepsilon dv = -\sum_j \sum_{i} B_i^T D B_j u_j dv$$
$i,j=1,2,.. N_{un}$

where $N_{un}$ stands for the number of unknown nodes. Discontinuity terms appeared in above equation but they cancelled by Eq. (3-7).

\[
K'_{ij} = \int_{V_i} B_i^T DB_j \, dv 
\]  
(3-30)

\[
- \sum_j \sum_e K'_{ij} = - \sum_e \int_{V_e} N_i (\nabla \cdot T^{th} + \nabla \cdot T^{ms} + \nabla \cdot T^{m} + \rho g e_i + \rho \ddot{u}) \, dv 
\]  
(3-31)

\[
\sum_e \int_{V_e} N_i (\nabla \cdot T^{th} + \nabla \cdot T^{ms} + \nabla \cdot T^{m}) \, dv = - \sum_e \int_{V_e} (T^{th} + T^{ms} + T^{m}) \nabla N_i \, dv = f^{th} + f^{ms} + f^{m}
\]

\[
\sum_e \int_{V_e} N_i (\rho g e_i) \, dv = W_i 
\]  
(3-32)

\[
f^{th}_i = - \sum_e \int_{V_e} T^{th}_i \nabla N_i \, dv 
\]  
(3-33)

\[
f^{ms}_i = - \sum_e \int_{V_e} T^{ms}_i \nabla N_i \, dv 
\]  
(3-34)

\[
f^{m}_i = - \sum_e \int_{V_e} T^{m}_i \nabla N_i \, dv .
\]  
(3-35)

Thus,

\[
\sum_e \sum_j (K'_{ij} u_j) = f_i ,
\]  
(3-36)

\[
i,j=1,2,.. n
\]

where $f$ is the resultant nodal force. In the matrix form:

\[
Ku = f ,
\]  
(3-37)

where $K$ is the global stiffness matrix.
For rotary devices such as motor, it is more intuitive to carry out the deformation analysis in the cylindrical coordinate system where the z-axis is the axis of rotation. To this end, the forces and displacements should be represented in the cylindrical coordinate system. The above derivations where based on the Cartesian coordinate system, thus the nodal forces are also in that system. Knowing the coordinates of the node in the Cartesian coordinates, we can convert the force and the displacement from the Cartesian to the cylindrical coordinate system using the following transformation:

\[ f = R \hat{f} \quad (3-38) \]
\[ u = R \hat{u} \quad (3-39) \]

where \( \hat{f} \) and \( \hat{u} \) are the equivalents of \( f \) and \( u \) in the cylindrical coordinate system respectively, and \( R \) is the rotation matrix around z-axis given by the following equation:

\[
R = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1 
\end{bmatrix}, \quad (3-40)
\]

where \( \theta = \arctan\left(\frac{y}{x}\right) \), with \( x \) and \( y \) being the coordinates of the node on \( x \) and \( y \) axes respectively. Substituting Eqs. (3-38) and (3-39) in Eq. (3-36) obtain:

\[
\sum_{\epsilon} \sum_{j} \left( K_{\epsilon}^j R_j \hat{u}_j \right) = R_i \hat{f}_i \quad (3-41)
\]

After pre-multiplying both sides by \( R_i^T \) we can write

\[
\sum_{\epsilon} \sum_{j} \left( \tilde{K}_{\epsilon}^j \hat{u}_j \right) = \hat{f}_i, \quad (3-42)
\]
where

\[ \tilde{K}_{ij} = R_i^T K_{ij}^e R_j, \] (3-43)

and finally by assembling the element matrices to form a global matrix, we have the static equation of equilibrium in the cylindrical coordinate system.

\[ \tilde{K}\mathbf{u} = \tilde{\mathbf{F}}. \] (3-44)

### 3.4 Calculation of Mechanical Stress in Magnetostrictive Media

It is important to note that if the dimensions change uniformly by magnetostriction the stress is still zero. This is verified by Eq. (2-120). The equation is written here again for convenience:

\[ \sigma = D\varepsilon + T^{ms}. \] (3-45)

If the model is free and model is uniformly magnetized, then \( \varepsilon = \varepsilon^{ms} \), and according to Eq. (2-119), \( T^{ms} = -D\varepsilon^{ms} \), thus the stress becomes zero. Fig. 3.2 shows the stress due to magnetostriction caused by boundary conditions.

![Fig. 3.2. Mechanical stress due to magnetostriction](image-url)
3.5 Calculation of Shrink-Fit and Press-Fit Stress Using Thermal Force

The construction process of motors includes a mounting process. A common and inexpensive way of mounting the stator in the housing is the heat to press approach, also known as *shrink-fitting*. The housing is fitted over the stator to retain it in position and to prevent it moving from the designed position as well as protecting the lamination. There are other methods too, such as gluing or press fitting. The shrink-fitting applies a high stress on the stator and the stress may alter the magnetic or magnetostriction property of the core. It may even change the mechanical property; therefore the stress must be calculated and considered in the calculations. Fig. 3.3 illustrates the fitting process and the radii change during the process. Initially, the stator outer radius $r_{s,\text{out}}$ is slightly larger than the housing inner radius $r_{h,\text{in}}$. The difference is called the radial interference $\delta$. Thus

$$\delta_r = r_{s,\text{out}} - r_{h,\text{in}}$$ (3-46)

The heat-up temperature $\Delta T = T_1 - T_2$ required for the fitting is proportional to $\delta$ according to the following equation:

$$r_{h,\text{in}} (1 + \alpha \Delta T) = r_{s,\text{out}}$$ (3-47)

where $\alpha$ is the coefficient of linear expansion. The above equations give

$$\Delta T = \frac{\delta_r}{\alpha r_{h,\text{in}}}$$ (3-48)
In linear elasticity, the shrink-fit stress is proportional to $\Delta T$; thus by knowing the level of required stress on the stator surface, we know $\Delta T$ and consequently $\delta_r$. The housing is then designed based on the value of $\delta_r$.

![Diagram of shrink-fitting process](image)

**Fig. 3.3. Shrink-fitting process**

The stress is considered to be zero at the moment when the radii become equal during the cool-down, in both stator and housing, and the finite element mesh is constructed based on the geometry at this moment. During the cooling process, the housing contracts till the final temperature. After cooling, the housing takes the thermal strain $\varepsilon^\text{th}$ given by the following equation:

$$
\varepsilon^\text{th} = \frac{-\delta_r}{r_{in}} \{1, 1, 1, 0, 0, 0\}. \tag{3-49}
$$

Note that the $\varepsilon^\text{th}$ is an internal strain and not an actual dimension change. We now define the thermal stress tensor $T^\text{th}$ as follows:

$$
T^\text{th} = -D\varepsilon^\text{th}. \tag{3-50}
$$
The minus sign indicates that $T^\text{th}$ is not the stress induced but the source of stress that causes $\varepsilon^\text{th}$, a concept similar to Eq. (2-119). The divergence of source stress tensor is equal to the volume force density, therefore, the nodal force method [27] can be used to calculate the equivalent thermal force:

$$ F^\text{th}_i = -\sum_{e} \int_{V_e} T^\text{th}_e \nabla N_i \, dv $$

(3-51)

After performing the matrix-vector multiplication in (3-50), $T^\text{th}$ are given by the following equations:

$$ T^\text{th}_{ij} = \frac{\delta_{er}}{r_{h,in}} E \frac{\delta_{ij}}{(1-2\nu)} $$

(3-52)

Once the force is known, the static equilibrium equation can be solved to find displacement:

$$ Ku = f^\text{th}(t) $$

(3-53)

Once $u$ is known, $\varepsilon$ can be calculated from Eq. (3-12). The stress in stator and housing is finally calculated using the following equation.

$$ \sigma = D\varepsilon - T^\text{th}. $$

(3-54)

Subtracting $T^\text{th}$ from the first term is to comply with the fact that the stress due to free and uniform thermal expansion/contraction is zero. $T^\text{th}$ is zero in the stator.
Chapter 4 Coupled Static Magneto-Mechanical Analysis

4.1 Introduction

Coupled magneto-mechanical problems are a class of problems in which magnetic and/or mechanical states are affected by each other. In an actuator, as an example, the magnetic flux density is affected by the actuator displacement. In a weakly coupled analysis, only the dependence of the mechanical state on the magnetic state is considered, whereas in strongly coupled problems, mutual dependence is taken into consideration. The mechanical state can affect the magnetic state through large deformations or via stress dependency of the magnetic permeability. We are concerned with the latter, which is referred to as the Villari effect (VE) [30] (also known as the inverse magnetostriction effect (IME) [13]). VE in electrical steel is small for practical levels of induced stress [28]; thus, in most electrical devices, a weakly coupled analysis yields valid results. However, giant magnetostrictive materials which are being used in transducers [23] and recently in interior permanent magnet (IPM) motors [31] show significant VE, demanding a strongly coupled analysis. In addition, when the material is subjected to large external loading, VE becomes significant even in the electrical steel. A number of coupled analysis methods have been proposed in the past [18]–[34]. Mohammed [19] used a set of magnetization curves under uniaxial stress to take into account VE while in works of Eason et al. [20], Delaere et al. [21], and also Besbes et al. [22],
VE is linked to MS strain and the coupled equations of the system are derived by minimizing the discretized functional of the system. In these methods, VE appears as an equivalent current term in the magnetic equation which cannot represent the stress-induced anisotropy. Moreover, the current term, unless strictly divergence-free, poses serious threat to the convergence of conjugate-gradient (CG) matrix solver when 3-D edge elements are used. Fonteyn et al. [24] derive the constitute relation of the material by minimizing the Helmholtz-free energy. The method, which gives the magnetic field as a function of the flux density and strain tensor, relies on a few parameters to be measured and its prediction of the magnetization is poor for low fields. Daniel [32] takes a different approach and describes not only VE, but also the magnetization of isotropic materials by homogenizing the poly-crystalline structure based on the magneto-elastic behavior of single crystals, using Boltzmann’s statistics. Both Belahcen’s and Daniel’s method predict the stress-induced anisotropy but due to the complexity of the domain structure and its evolution in iron, the methods show significant discrepancies with the measurement, in particular at low fields.

Following Besbes et al. [22], the authors of this paper have already proposed a method in which the measured MS characteristic is used for the prediction of VE [33], however, the method does not predict the stress-induced anisotropy. In this paper, the method is extended to predict the stress-induced anisotropy. Unlike [24] and [32], the proposed method does not try to model the material, but rather uses the measured MS for the prediction of VE.
4.2 Coupled Analysis under Longitudinal Stress

By defining the magnetic field strength \( H(B,\sigma) \) as the partial derivative of the total stored energy density of the coupled magneto-mechanical system with respect to \( B \) [8] and \( H_0 = H(B,0) \), the energy functional of the system in the static case, assuming that no external force is doing work on the system; can be defined as follows [8]:

\[
E_t = \int_{\Omega} e_{\text{mag}} dv + \int_{V} e_{\text{mech}} dv ,
\]

\[
e_{\text{mag}} = \int_{0}^{R} \mathbf{H}' \cdot dB' - \mathbf{Q} \cdot \mathbf{B} ,
\]

\[
e_{\text{mech}} = \frac{1}{2} \mathbf{\sigma} \cdot \mathbf{\varepsilon}^{\text{elast}} ,
\]

where \( E_t \) is the total energy of the system, \( e_{\text{mag}} \) and \( e_{\text{mech}} \) are the magnetic and mechanical energy densities, \( J \) is the source current density, \( A \) is the magnetic vector potential, and \( \Omega \) and \( V \) are the magnetic and mechanical domains, respectively.

It can be demonstrated that

\[
\int_{\Omega} \mathbf{J} \cdot \mathbf{A} dv = \int_{\Omega} \mathbf{Q} \cdot \mathbf{B} dv ,
\]

where \( \mathbf{Q} \) is the current vector potential. Eq. (4-2) then becomes

\[
e_{\text{mag}} = \int_{0}^{R} \mathbf{H}' \cdot dB' - \mathbf{Q} \cdot \mathbf{B} .
\]

The stationary point of \( E_t \) is the solution of the system. The underlying variables
of the system are the displacement field \( u \) and the magnetic flux \( \phi \):

\[
\left. \delta E \right|_{u=const} = 0, \quad (4-6)
\]

\[
\left. \delta E \right|_{\phi=const} = 0. \quad (4-7)
\]

Eq. (4-6) can be written as follows

\[
\int_{\Omega} \delta (e_{\text{mag}} + e_{\text{mech}}) \left|_{u} \right. = 0. \quad (4-8)
\]

Note that we have extended the mechanical domain to \( \Omega \) and it costs us nothing because the mechanical energy is zero outside \( V \). For the first term we have

\[
\left. \delta e_{\text{mag}} \right|_{u} = (H_0 - Q) \cdot \delta B. \quad (4-9)
\]

Constant \( u \) means constant \( \varepsilon \), hence \( \delta e^{\text{elast}} = -\delta e^{\text{ms}} \). Thus

\[
\left. \delta e_{\text{mech}} \right|_{u} = \sigma \cdot \delta e^{\text{elast}} = -\sigma \cdot \delta e^{\text{ms}} = -\sigma \cdot S \delta \lambda - \lambda \sigma \cdot \delta S. \quad (4-10)
\]

Here we assume that the stress does not change the direction of \( B \), so the last term vanishes. Expanding \( \delta \lambda \) we have

\[
\delta \lambda = \left( \frac{\partial \lambda}{\partial B} + \frac{\partial \lambda}{\partial \sigma_B} \frac{\partial \sigma_B}{\partial B} \right) \cdot \delta B, \quad (4-11)
\]

\[
\frac{\partial \lambda}{\partial B} = \frac{1}{B} \frac{\partial \lambda}{\partial B} B. \quad (4-12)
\]

It can be shown that \( \frac{\partial \sigma_B}{\partial B} = -\frac{E}{(1 + \nu_p)} \frac{\partial \lambda}{\partial B} \cdot \delta B \). Eq. (4-10) then becomes

\[
\left. \delta e_{\text{mech}} \right|_{u} = \nu_{\text{ms}} B \cdot \delta B. \quad (4-13)
\]
where $\nu_{ms}$, given by the following equation, is referred to as the MS reluctivity in this paper:

$$\nu_{ms} = -\frac{1}{B} \left( 1 - \frac{E}{1 + \nu_p} \right) \frac{\partial \lambda}{\partial \sigma} \cdot S .$$  \hspace{1cm} (4-14)

Eq. (4-8) becomes

$$\int_{\Omega} (H_0 - Q + \nu_{ms} B) \cdot dB \, dv = 0 , \hspace{1cm} (4-15)$$

Using the fundamental lemma of the calculus of variation, the following equation is obtained from (4-15):

$$H_0 + \nu_{ms} B = Q . \hspace{1cm} (4-16)$$

Taking the curl of both sides we obtain Ampere’s law as the energies were defined based on Ampere’s law.

$$\nabla \times H = J , \hspace{1cm} (4-17)$$

$$H = H(B, \sigma) = H_0 + \nu_{ms} B . \hspace{1cm} (4-18)$$

Substituting $H_0 = \nu(B,0)B$ and $B = \nabla \times A$, we obtain the magnetic equation of the coupled system in term of magnetic vector potential:

$$\nabla \times ((\nu + \nu_{ms})\nabla \times A) = J , \hspace{1cm} (4-19)$$

where $\nu = \nu(B,0)$ is the magnetic reluctivity evaluated from the stress-free magnetization curve, and $\nu_{ms}$ is the coupling term representing the variation of reluctivity due to mechanical stress. Measurement would give the sum of the two reluctance terms in (4-19) therefore Eq. (4-18) in fact calculates the magnetization curves under stress.

The finite element discretization of (4-19) using the Galerkin weighted residual
method yields:

\[ MA = P, \quad (4-20) \]

where \( M \) is the magnetic coefficient matrix, \( P \) is the current source term. The discretized static mechanical equation has the following form

\[ Ku = f^m + f^{ms}, \quad (4-21) \]

where \( K \) is the global stiffness matrix, and \( f^m \) and \( f^{ms} \) are the loading vectors of nodal magnetic and MS forces, respectively.

Eqs. (4-20) and (4-21) are coupled and nonlinear demanding a coupled analysis algorithm. In this work, we adopted the indirectly coupled method outlined in [33].

---

**Fig. 4.1. Indirectly strong coupling procedure**

- \( n \): Newton-Raphson iteration number
- \( k \): fixed-point iteration number
- conv.: convergence

- **Initialization**: \( \nu_{ms}^0 = 0 \)
- **Calculating \( B \) (linear analysis)**
- **Calculating \( B \) (nonlinear analysis)**
- **Newton-Raphson loop**
- **Fixed-point loop**

---

\( u^k, \varepsilon^k, \sigma^k \) : structural analysis

\( F^{m,k}, F^{ms,k} \) : equilibrium forces in magnetic and MS elements, respectively

\( T^{m,k}, T^{ms,k} \) : magnetic and MS stress tensors, respectively

conv.: convergence
4.2.1 Numerical Results

Using Eq. (4-14) in one dimension, two magnetization curves for stress levels of -5.23 MPa and 5.23 MPa were calculated. Fig. 4.3-(a) and (b) show the measured stress-free magnetization curve and a set of magnetostriction curves measured for isotropic non-oriented electrical steel [35] and the calculated curves are shown in Fig. 4.4. As can be seen, for the range of flux density shown in the graph, both compressive and tensile stress cause an increase in the reluctivity and this is consistent with the theory because in both cases stress and λ have opposite signs, increasing the micromagnetic potential energy and resulting a lower magnetization [35]. However validation of the curves requires accurate measurement of the curves which will be perused in future.

The proposed model was then applied to a simple 2-D problem shown in Fig. 4.2.

![Fig. 4.2. Analyzed model and the analysis condition.](image)
An uncoupled analysis was carried out first to obtain the stress with the convergence criterion of 0.001 T for the nonlinear analysis. Fig. 4.5-(a) shows the flux density distribution. Fig. 4.5-(c), (d), and (e) show the three stress components respectively. The main cause of the stress is the magnetic force due to the air gap and the effect of MS on was on the stress was insignificant.

Comparing the calculated stress against the calculated curves, one does not expect a significant modification of the flux density distribution by stress; nevertheless we performed a coupled analysis of the problem according to the flowchart in Fig. 4.1 with convergence criterion of 0.001 T for both loops. Fig. 4.5-(b) shows the flux density distribution obtained from the coupled analysis. The stress has slightly modified the flux density distribution in the left leg of the core. Fig. 4.6 compares the $y$-component of the flux density in the left leg along line $y=0$ for these analysis: uncoupled, coupled with the proposed model, and coupled based on the traditional method of using a set of magnetization curves as in [31] and [35]. However the curves were calculated using Eq. (4-17) in one-dimension (1D) and for thirty levels of stress linearly distributed between -5.23 MPa and 5.23 MPa. The error between the two coupled cases is due to the fact that the curves were calculated in 1D whereas in the proposed method, $\nu_{ms}$ is calculated from the 3-D stress tensor. In plane strain analysis we have $\sigma_x = \nu_p (\sigma_{xx} + \sigma_{yy})$. 
Fig. 4.3. (a) Experimental magnetization curve, (b) magnetostriction curves of the material used in the analysis.

Fig. 4.4. Stress-free and two calculated magnetization curves
Fig. 4.5. (a) and (b) flux density distribution obtained from uncoupled and coupled analysis using the proposed model, (c), (d) and (e) stress components $\sigma_{xx}$, $\sigma_{yy}$, and $\sigma_{xy}$, respectively.

Fig. 4.6. $y$-component of flux density in the left leg of the reactor along line $y=0$, obtained from three analysis.
4.3 Coupled Analysis under Multi-Axial Stress

A magneto-mechanical system is described by its magnetic and mechanical state variables. One can choose the magnetic flux $B$ and mechanical strain tensor $\varepsilon$ as state variables, and express the magnetic field $H$ and stress tensor $\sigma$ by constitutive laws of the material in terms of the state variables:

$$H = H(B, \varepsilon), \text{ and } \sigma = \sigma(B, \varepsilon),$$

where $\varepsilon$ is linked to deformation $u$ by geometric law [22]:

$$\varepsilon = \frac{1}{2} (\nabla u + (\nabla u)^T).$$ (4-22)

The Ampère law is valid in magnetostrictive media too:

$$\nabla \times H(B, \varepsilon) = J$$ (4-23)

where $j$ is the current density. By multiplying both sides of (4-23) by $dA$ (an arbitrary variation of magnetic vector potential $A$) and integrating over the whole magnetic domain we have
\[ \int \nabla \times \mathbf{H}(\mathbf{B}, \varepsilon) \cdot dA = \int \mathbf{J} \cdot dA. \] (4-24)

The right-hand-side term of (4-24) is the variation of energy injected to the system by the current source. If the variation is under the constraint of constant \( \varepsilon \), the energy is equal to \( w \), the sum of stored magnetic and elastic energies. Using proper vector identity together with divergence theorem, the left-hand-side is simplified and the following equation is obtained:

\[ \int_{\omega} \mathbf{H}(\mathbf{B}, \varepsilon) \cdot d\mathbf{B} = \int_{\omega} \delta w(\mathbf{B}, \varepsilon) d\mathbf{v}. \] (4-25)

which reveals the following equation between the \( h \) and \( w \):

\[ \mathbf{H}(\mathbf{B}, \varepsilon) = \frac{\partial w(\mathbf{B}, \varepsilon)}{\partial \mathbf{B}}. \] (4-26)

An alternative, which is preferred in this paper, is to choose \( \mathbf{H} \) and \( \sigma \) as state variables. In this case \( \mathbf{B} = \mathbf{B}(\mathbf{H}, \sigma) \), and \( \varepsilon = \varepsilon(\mathbf{H}, \sigma) \).

With a similar procedure one would arrive at:

\[ \mathbf{B}(\mathbf{H}, \sigma) = \frac{\partial \omega_\varepsilon}{\partial \mathbf{H}} \] (4-27)

where \( \omega_\varepsilon \) is the co-energy (complementary energy) of the system which can be written in the following form [22].

\[ \omega_\varepsilon(\mathbf{H}, \sigma) = \int_{0}^{\mathbf{H}} \mathbf{B}'(\mathbf{H}', 0) \cdot d\mathbf{H}' + \int_{0}^{\sigma} \varepsilon'(\mathbf{H}, \sigma') : d\sigma' \] (4-28)

where the two-dot product is defined (using Einstein summation notation) as follows:

\[ \varepsilon : \sigma = \varepsilon_{ij} \sigma_{ij}. \] (4-29)
Note that the stress-free B-H curve is used in (4-28) and, luckily, the B-H curves available for materials are usually under zero stress. This is the reason behind our choice of $H$ and $\sigma$ as the state variables.

The strain tensor in (4-28) is the sum of the magnetostriction strain tensor $\varepsilon^{\text{ms}}$ and the excess strain tensor $\varepsilon^{\text{ex}}$ [33].

$$\varepsilon(H, \sigma) = \varepsilon^{\text{ms}}(H, \sigma) + \varepsilon^{\text{ex}}(\sigma).$$

(4-30)

The excess strain is the strain due to other sources than magnetostriction and is related to the stress tensor through the Hooke’s law [21], [22]:

$$\varepsilon^{\text{ex}} = D\varepsilon^{\text{ex}}$$

(4-31)

where $D$ is the tensor of elasticity, which is constant in linearly elastic materials. Thus, restricting the discussion to linear elasticity, we have

$$w_{\varepsilon}(H, \sigma) = \int_{0}^{H} B'(H', 0) \cdot dH' + \frac{1}{2} \sigma : \varepsilon^{\text{ex}} + \sigma : \varepsilon^{\text{ms}},$$

(4-32)

therefore

$$B(H, \sigma) = B(H, 0) + \sigma : \frac{\partial \varepsilon^{\text{ms}}}{\partial H}.$$  (4-33)

Assuming an isotropic and isochoric MS, $\varepsilon^{\text{ms}}$ is given by the following equations [33]:

$$\varepsilon^{\text{ms}} = \lambda S$$

(4-34)

$$S_{ij} = \left( \frac{3}{2} \frac{H_i H_j}{H^2} - \frac{1}{2} \delta_{ij} \right), \quad i, j = 0, 1, 2$$

(4-35)
where $\lambda$ is the magnetostriction parameter, $S$ is the MS direction tensor, and $\delta_{ij}$ is Kronecker’s delta. Eq. (4-31) can be written in the following form:

$$\sigma = D(\varepsilon - \varepsilon^{ms}) = D\varepsilon - \sigma^{ms}, \quad (4-36)$$

with

$$\sigma^{ms} = D\varepsilon^{ms} = \frac{E}{1 + \nu_p} S, \quad (4-37)$$

where $\sigma^{ms}$ is the induced MS stress, and $E$ and $\nu_p$ are Young’s modulus, and Poisson’s ratio, respectively. $\sigma^{ms}$ can be thought of as a stress induced by a field-dependent local source of stress, similar to the Maxwell stress tensor, which balances with $\sigma^{ms}$ [33]:

$$T^{ms} = -\sigma^{ms} = \frac{-E}{1 + \nu_p} S \quad (4-38)$$

where $T^{ms}$ is the MS stress tensor discussed in Section 2.9. The strain $\varepsilon$ depends on $\sigma$ and $H$. If MS is neglected, the first term on the right-hand-side of (4-30) and the second term on the right-hand-side of (4-33) vanish, leaving $B$ and $\varepsilon$ independent of $\sigma$ and $H$, respectively (See Fig. 4.7).

From (4-33) and (4-34) we have

$$B(H, \sigma) = B(H, 0) + \left( \sigma : S \frac{\partial \lambda}{\partial H} + \lambda \sigma : \frac{\partial S}{\partial H} \right), \quad (4-39)$$

The first term in the parentheses can be written as the products of a scalar and $H$, whereas the second term becomes the product of a tensor and $H$; thus

$$B(H, \sigma) = B(H, 0) + (\mu_1^{ms} + \mu_2^{ms})H, \quad (4-40)$$

where

$$\mu_1^{ms} = \frac{\sigma : S}{H} \left. \frac{\partial \lambda}{\partial H} \right., \quad (4-41)$$
\[ \mu_2^{ms} = \frac{3\lambda}{H^2} (\sigma - \sigma_i I), \]  
(4-42)

where \( H \) is the magnitude of \( H \), \( \sigma_i \) is the longitudinal stress, given by (4-43) and \( I \) is the identity tensor.

\[ \sigma_i = \hat{h}^T \sigma \hat{h}, \]  
(4-43)

where \( \hat{h} \) is the unit vector in \( H \) direction. Thus

\[ \mu(H, \sigma) = [\mu(H, 0) + \mu_1^{ms}(H, \sigma)] I + \mu_2^{ms}(H, \sigma). \]  
(4-44)

The total permeability can be written in the following form:

\[ \mu(H, \sigma) = \begin{bmatrix} \mu_s & 0 & 0 \\ 0 & \mu_s & 0 \\ 0 & 0 & \mu_s \end{bmatrix} + \frac{3\lambda}{H^2} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}, \]  
(4-45)

where \( \mu_s \) is given by the following equation:

\[ \mu_s = \mu(H, 0) + \sigma : S \frac{\partial \lambda}{\partial H} - \frac{3\lambda}{H^2} \sigma_i. \]  
(4-46)

The double-dot product in (5-39) can be written in a more instructive form:

\[ \sigma : S = \sigma_{eq} = \frac{3}{2} \hat{h}^T \left\{ \sigma - \frac{1}{3} tr(\sigma) I \right\} \hat{h}, \]  
(4-47)

Where \( \sigma_{eq} \) denotes the equivalent stress, \( tr(.) \) stands for the trace of a tensor and the term in the brackets is commonly known as the deviatoric part if the stress tensor [61]. It can be verified that for hydrostatic pressure both \( \mu_1^{ms} \) and \( \mu_2^{ms} \) are zero, and this is the result of assuming MS to be isochoric. Eq. (4-47) was used Eq. in (2-126).

The stress dependency of \( \mu_1^{ms} \) and \( \mu_2^{ms} \) is not only through \( \sigma \) but also through \( \lambda \) which depends on \( \sigma \). When the measurement data for \( \lambda \) is uni-axial, we use the longitudinal stress \( \sigma_{eq} \), given by Eq. (4-43) for its evaluation. When multi-axial
measurement is available for $\lambda$, the method can readily take it into account through the evaluation of $\lambda$ using the transverse stress as well as the longitudinal stress.

In the magnetostatic case, the Ampère law now becomes

$$\nabla \times (\nu(H, \sigma) \nabla \times A) = J,$$

(4-48)

where $\nu(H, \sigma) = \mu^{-1}(H, \sigma)$ is the magnetic reluctivity, and $A$ and $J$ are the magnetic vector potential and current density, respectively.

The finite element discretization of (4-48) becomes

$$M(a, \sigma)a = j,$$

(5-49)

where $M$ is the magnetic coefficient matrix, $a$ is the solution vector (the circulation of $A$ along edges in 3-D case), and $j$ is the current source term. The Newton-Raphson linearization of (4-49) takes the following form:

$$M \left[ \nabla \nu \right] a \Delta a = j - M a,$$

(5-50)

where $k$ is the iteration number. However, in the proposed model, due to the extra terms in the reluctivity, the derivative of $\nu$ with respect to $a$ includes new terms arising from the second and third terms on the right-hand-side of (4-44). Using the indirectly coupled procedure [13], the stress remains constant during the Newton-Raphson iteration; thus $\sigma$ is treated as a constant in the calculation of the derivative. The following equation serves as a guide to the calculation of the derivative:

$$\frac{\partial \nu}{\partial a} = \frac{\partial \mu^{-1}}{\partial a} = \mu^{-1} \frac{\partial \mu}{\partial a} \mu^{-1}.$$  

(4-51)
4.3.1 Numerical Results

4.3.1.1 Prediction of B-H Curves under Uni-Axial Stress

The prediction of the model is first investigated by calculating two B-H curves of pure iron under compressive and tensile stresses of -11.1 MPa and 10.1 MPa, respectively. In this case $\mu_2^m$ has no effect on the calculated curves. The used data are three magnetostriction curves under -11.1 MPa, 0.0 MPa, and 10.1 MPa, respectively, the stress-free B-H curve available from the literature [36]. The used data and the calculated curves are shown in Fig. 4.9. When $\lambda$ curves reach their maximum, their derivative with respect to h becomes zero, making the corresponding B-H curves cross over the stress-free curve (the Villari reversal [35]). At the cross over points $\mu_2^m$ reaches its maximum but it is negligible compared with the stress-free permeability because this happens near saturation; thus VE becomes effective only in low field regions.

Fig. 4.8. (a) Magnetostriction curves used as the input data, (b) calculated B-H curves for -11.1 and 10.1 MPa using the proposed method.
4.3.1.2  Effect of Stress on Distribution Of Magnetic Flux Density

For the case of multi-axial loading, the stress-induced anisotropy can be illustrated by calculating the flux distribution using a coupled analysis and observing the angle between b and h. To this end, the method is applied to a simple 2-D model described in Fig. 4.9. The stress distribution within the core due to the applied stress is shown in Fig. 4.10. In this certain problem, the effect of the magnetic and magnetostriction forces on the stress is negligible.

Fig. 4.9. Analyzed 2-D problem.

The result of the coupled analysis of the problem using the proposed method is shown in Fig. 4.11. The flux density obtained from the uncoupled analysis is shown in Fig. 4.11-a while Fig. 4.11-b show the distribution obtained from the coupled analysis neglecting $\mu^m_r$. Finally the effect of $\mu^m_r$ can be seen in Fig. 4.11-c where both permeability terms have been taken into the analysis. In Fig. 4.11-c, not only the direction but the magnitude of the flux has been affected by the stress.
Fig. 4.10. Stress distribution within the core due to the applied stress.

Fig. 4.11. Flux density distribution; (a) uncoupled analysis, (b) coupled analysis neglecting $\mu_2''$, (c) coupled analysis considering both $\mu_1'''$ and $\mu_2'''$. Blue arrows in (c) indicate the direction of the magnetic field.
4.4 Summary

In this chapter, we presented a model for the coupled analysis of magneto-mechanical systems taking account of magnetostriction and its inverse effect. The coupled equations of the continuum system were derived using the calculus of variation. In the proposed model, the permeability variation due to stress is calculated from the stress tensor and the magnetostriction curves measured under stress. The finite element formulation for the indirectly strong coupling procedure was presented and the stress and flux density distribution distributions in a simple reactor model were calculated.

We extended our proposed method to describe the effect of multi-axial stress on the magnetic permeability when only uni-axial measurement equipment is available. Uni-axial B-H curves cannot describe the Villari effect under multi-axial stress, especially the stress-induced anisotropy, but we showed that uni-axial magnetostriction curves can be used to predict the variation in permeability due to multi-axial stress. Moreover, if the magnetostriction data measured under multi-axial stress is available, the model predicts the Villari effect more accurately. The permeability variation due to stress is given explicitly, making the implementation straightforward by a minor modification of the Newton-Raphson method.
Chapter 5  Dynamic Structural Analysis

5.1 Introduction

In Chapter 3, the discretized static equation of structures was derived. The static equation which is repeated here:

\[ Ku = f, \quad (5-1) \]

Where \( K, u \) and \( f \) are the stiffness matrix, the displacement vector and the loading term respectively. Eq. (5-1) is applicable when the loading is time invariant but can also be safely used under quasi-static loading in which the inertial and frictional forces are negligible. In this case, at each instance of time, the static equation can be used to find the solution at the specific instance of time and this makes the solution also quasi-static.

Being quasi-static is a relative feature of the system. To be quantitative, if the highest significant time harmonic of the force is well below the lowest eigen-frequency of the system, the loading is quasi-static. One example of quasi-static loading is the defamation under thermal expansion. The temperature may increase with time but the variation is often slow with respect to the system.

Electrical machine are often dynamic and the loading variation is often with harmonics of excitation source which is usually over 50 Hz. Devices which are fed driven by inverter power supply have harmonics of several thousands of Hz in addition to the base frequency of the excitation source. Nonlinearity of the cores also induce harmonics in the magnetic field and thus in the loading term. Furthermore, due to existence of the coil which is loosely connected to the core and
also due to the insulators between steel laminates, the structure often has a relatively large friction term. Therefore it is often necessary to consider the inertial and friction terms in the analysis.

In the following section, the derivations in Chapter 3 are extended to take account of the inertial and friction term. This results in a second order differential equation of motion whose solution demands a more attention than in the static case.

5.2 Equation of Equilibrium Including Friction and Inertia Terms

In Chapter 3, the finite element equations of static structures were derived. In the dynamic case, the inertial and friction (damping) terms are also present in the total volume force density:

\[ m \ddot{u} + \rho \dddot{u} + \rho \dot{u} + f_v \]

where \( \dddot{u} \) and \( \dot{u} \) are inertial and friction terms of the total force density.

5.3 Finite Element Discretization of Dynamic Equilibrium Equation

Following the same procedure as in Chapter 3, the following equations are obtained:
\[
\sum_{\varepsilon} \left[ N_e \left( \rho g e_{\varepsilon} + c \sum_j N_j u_j + \rho \sum_j N_j \ddot{u}_j \right) \right] dv = W_i + \sum_{\varepsilon} \sum_j M_{\varepsilon j} \ddot{u}_j 
\]

(5-3)

\[
M_{\varepsilon j} = \int_{V_e} \rho^e w_{\varepsilon j} dv 
\]

(5-4)

\[
C_{\varepsilon j} = \int_{V_e} c^e w_{\varepsilon j} dv 
\]

(5-5)

\[
w_{\varepsilon j}^{(ic)} = \begin{bmatrix} N_i N_j & 0 \\ 0 & N_i N_j \end{bmatrix} 
\]

(5-6)

Thus,

\[
\sum_{\varepsilon} \sum_j \left( K_{\varepsilon j} u_j + C_{\varepsilon j} \ddot{u}_j + M_{\varepsilon j} \dddot{u}_j \right) = f_i , 
\]

(5-7)

\[i,j=1,2,...,n.\]

In the matrix form:

\[
M\ddot{u} + C\dddot{u} + K\dot{u} = f 
\]

(5-8)

Similar to the derivation in Chapter 3, the cylindrical coordinate system we obtain:

\[
\sum_{\varepsilon} \sum_j \left( \tilde{K}_{\varepsilon j} \dddot{u}_j + \tilde{C}_{\varepsilon j} \dddot{u}_j + \tilde{M}_{\varepsilon j} \dddot{u}_j \right) = \hat{f}_i , 
\]

(5-9)

where

\[
\tilde{K}_{\varepsilon j} = R^T \tilde{K}_{\varepsilon} R_j , 
\]

(5-10)

\[
\tilde{C}_{\varepsilon j} = R^T \tilde{C}_{\varepsilon} R_j , 
\]

(5-11)

\[
\tilde{M}_{\varepsilon j} = R^T M_{\varepsilon} R_j . 
\]

(5-12)

Finally by assembling the element matrices into global matrices we have the equation of motion in cylindrical coordinate system.
\[ \ddot{\mathbf{u}} + \dot{\mathbf{C}} \mathbf{u} + \mathbf{K} \mathbf{u} = \mathbf{f}. \]  (5-13)

5.4 Solution Methods of Motion Equation

Finite element space discretization of magneto-mechanical systems often results in a first order magnetic differential equation, coupled with a second order motion equation [17]. The magnetic equation which usually includes material nonlinearity has to be solved using a proper time integration method to obtain the magnetic flux density and consequently the magnetic forces. The motion equation is often linear and one way-coupled, so it can be solved using the modal decomposition method [46], [47] or a time integration method [37-39]. When the steady-state solution of the motion equation is sought, the frequency domain method [17] is also another choice. However, when motion equation is nonlinear or when it is two-way coupled with the magnetic equation, through large displacements for example, the modal and frequency domain methods are no longer applicable and one has to resort to time integration methods [38], [41].

The time domain, frequency domain and modal decomposition methods are essentially different approaches but they yield very similar results. However they have different performances regarding computation time and accuracy and the right choice of method for the problem needs a deep understanding of the methods.

In this paper we discuss the three methods with the focus of the discussion on the modal decomposition method as it is often the fastest but the implementation is much more complicated.
5.4.1 Time Domain Methods

The motion equation is often linear, and one way-coupled, so it can be solved using the modal decomposition method [A-8] or a time integration method [2-4]. When the steady-state solution of the motion equation is sought, the frequency domain method [17] is also another choice. However, when the equation is nonlinear or when it is coupled with the magnetic equation, through large displacements for example, the modal and frequency domain methods are no longer applicable and one has to resort to time integration methods [38], [41].

Time integration methods (TIMs) are numerical methods for solving differential equation by integrating over time. Among the many available methods, some may not fit certain class of problems due to stability issues. Moreover, some TIMs are superior to others regarding the accuracy and/or computation costs.

Recall the equation of motion derived in previous section:

\[ M\ddot{u} + C\dot{u} + Ku = f \quad (5-14) \]

A TIM starts by first discretizing the time interval of the analysis into time steps \( \Delta t \). The time steps do not need to be equal although it is often better to be. A TIM has to be stable otherwise the error grows large and arithmetic overflow of computer occurs [43]. The stability can be investigated quantitatively by first writing the relation between two successive solutions \( x_n \) and \( x_{n-1} \) as follows:

\[ x_n = Ax_{n-1} + Bf_n, \quad (5-15) \]
where $A$ and $B$ are matrices which depend on step size $\Delta t$ and if nonlinear on $x_n$. With $f_n=0$, we have $x_n=A^n x_0$. In order to have bounded solution, the following conditions must be met:

$$|\rho(A)| \leq 1,$$  \hspace{1cm} (5-16)

where $\rho(A)$ is spectral radius of $A$ which is equal to the largest eigenvalue of $A$.

The order of accuracy of a TIM is said to be $q$ (or the method is said to be $q_{th}$-order accurate) if the local truncation error, i.e. the error caused by one iteration, is proportional to $(\Delta t)^q$.

5.4.1.1 Backward Euler Method

Without losing the generality of the method, we assume that the time step $\Delta t$ is equal for all the steps. Eq. (5-14) can be discretized straightforwardly using the Euler method:

$$\dot{u}_{t+\Delta t} = \frac{1}{\Delta t} (u_{t+\Delta t} - u_t)$$  \hspace{1cm} (5-17.a)

$$\dot{u}_{t+\Delta t} = \frac{1}{\Delta t^2} [u_{t+\Delta t} - 2u_t + u_{t-\Delta t}]$$  \hspace{1cm} (5-17.b)

where $u_t$ and $u_{t-\Delta t}$ are known values. Thus

$$K^* u_{t+\Delta t} = \Delta t^2 f_{t+\Delta t} - 2\Delta t C u_t + \Delta t^2 K u_{t-\Delta t},$$  \hspace{1cm} (5-18)

where

$$K^* = M + \Delta t C + \Delta t^2 K.$$  \hspace{1cm} (5-19)

$K^*$ is symmetric and positive-definite, therefore it can be solved by incomplete
conjugate gradient (ICCG) method. By using the solution of each step as the initial value of the ICCG method in the next step, much faster convergence is achieved specially when $\Delta t$ is small and hence the successive solutions differ slightly.

The method is not self-starting and needs the values of $u$ at two previous time-steps. A reasonable choice is to use the static solutions for the initial steps. In addition, the method has one order of accuracy and introduces numerical damping, i.e. if the structure is given an initial value and with zero force, and then the constraint is removed the solution fades although it should keep oscillating according to analytic solution. The numerical damping decreases by choosing smaller $\Delta t$ and its effect of the steady state solution is not very significant but when transient solution is sought, the numerical damping may become intolerable. Mechanical and civil engineers often use an alternative method which does not entail the damping problem and has a higher accuracy without significant extra cost. The method is called the Newmark method which is explained below. We too use the Newmark method in this paper.
Fig. 5.1. Comparison of solution of mass-spring system under force of \( f = \sin(2\pi t) + 0.5\sin(4\pi t) \), with two step size. Spring constant is 10 N/m, mass is 0.1 kg., the system is initially at rest.

5.4.1.2 Newmark Method

The Newmark method specifically developed for the second order motion equation can be described in the form of the truncated Taylor’s series of \( u \) and \( \dot{u} \) as follows [9]:

\[
\begin{align*}
\mathbf{u}_{t+\Delta t} &= \mathbf{u}_t + \Delta t \dot{\mathbf{u}}_t + \frac{1}{2} \Delta t^2 \ddot{\mathbf{u}}_t + \beta \Delta t^3 \dot{\mathbf{u}}_t , \\
\dot{\mathbf{u}}_{t+\Delta t} &= \dot{\mathbf{u}}_t + \Delta t \ddot{\mathbf{u}}_t + \gamma \Delta t^2 \dot{\mathbf{u}}_t ,
\end{align*}
\]

where \( \beta \) and \( \gamma \) are the parameters of the method. Note that the third derivatives in (5-20.a) and (5-20.b) have been replaced with \( \dddot{\mathbf{u}}_t \), given by (5-21), multiplied by \( 4\beta \) and \( 2\gamma \) respectively.
\[ \ddot{u} = \frac{1}{\Delta t} (\ddot{u}_{t+\Delta t} - \ddot{u}_t). \]  

(5-21)

By substituting (5-21) in (5-20.a) and (5-20.b), the following equations are obtained:

\[ \ddot{u}_{t+\Delta t} = b_1 (u_{t+\Delta t} - u_t) + b_2 \dot{u}_t + b_3 \ddot{u}_t, \]  

(5-22.a)

\[ \ddot{u}_{t+\Delta t} = b_4 (u_{t+\Delta t} - u_t) + b_5 \dot{u}_t + b_6 \ddot{u}_t, \]  

(5-22.b)

\[ b_1 = \frac{1}{\beta \Delta t^2}, b_2 = -\frac{1}{\beta \Delta t}, b_3 = 1 - \frac{1}{2\beta}, b_4 = \gamma \Delta t b_1, b_5 = 1 + \gamma \Delta t b_2, b_6 = \Delta t (1 + \gamma b_3 - \gamma) \]

The unknown \( u_{t+\Delta t} \) is found by substituting (5-22.a) and (5-22.b) in (5-14) and solving the resultant equation. The stability condition of the Newmark method for undamped linear motion equation is [15]:

\[ \Delta t < \frac{1}{\omega_{\text{max}} \sqrt{\gamma/2 - \beta}}. \]  

(5-23)

With \( \beta = 1/4 \) and \( \gamma = 1/2 \), the method is second-order accurate and it is unconditionally stable. The computation cost of each step is nearly the same as in the Euler method; hence the method is very popular in structural dynamic analysis.

5.4.2 Frequency Domain Method

In the vibration analysis, we are not concerned with the transient but the steady-state response which is a periodic solution due to the periodic loading. The frequency domain method is applicable to linear motion equation to solve for the steady state solution. In this method, the loading with period \( T \) is first written in its Fourier representation:
\[ f(t) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{j\omega_0 t}, \]  
(5-24)

where \( \omega_0 = \frac{2\pi}{T} \), and \( \hat{f}_k \) are the Fourier coefficients given by the following equation:

\[ \hat{f}_k = \frac{1}{T} \int_0^T f(t) e^{-j\omega_0 t} \, dt, \]  
(5-25)

Upon discretization, the integral in (11) is replaced with the following summation:

\[ \hat{f}_k = \frac{1}{N} \sum_{n=0}^{N-1} f_n e^{-j\Omega_n}, \]  
(5-26)

where \( \Omega_n = \frac{2\pi}{N} \), and \( N \) is the number of time steps in one period \( (T = N\Delta t) \). It is necessary to use a constant \( \Delta t \) in this method. Multiplying both sides of (5-25) by \( e^{j\Omega_n m} \) and summing from 0 to \( N-1 \), we obtain the following equation

\[ f_n = \sum_{k=0}^{N-1} \hat{f}_k e^{j\Omega_n}. \]  
(5-27)

The following relation was used in the derivation of (5-27).

\[ \sum_{k=0}^{N-1} e^{j\Omega_n(m-n)} = \begin{cases} N & m = n \\ 0 & \text{else} \end{cases} \]  
(5-28)

It is more common to write (5-27) in the following form:

\[ f_n = \frac{1}{N} \sum_{k=0}^{N-1} F_k e^{j\Omega_n}, \]  
(5-29)

where \( F_k = N\hat{f}_k \) is referred to as the discrete Fourier transform of \( f \) defined as follows:

\[ F_k = \sum_{n=0}^{N-1} f_n e^{-j\Omega_n} \]  
(5-30)

Similarly we have
\[ u_n = \frac{1}{N} \sum_{k=0}^{N-1} U_k e^{j\Omega_n k} \]  
\( (5-31) \)

It can be shown that

\[ \dot{u}_n = \frac{1}{N} \sum_{k=0}^{N-1} jk\omega_0 U_k e^{j\Omega_n k}, \]  
\( (5-32) \)

\[ \ddot{u}_n = -\frac{1}{N} \sum_{k=0}^{N-1} \alpha_k^2 k^2 U_k e^{j\Omega_n k}. \]  
\( (5-33) \)

Substituting the above equations in motion equation we obtain.

\[ \sum_{k=0}^{N-1} (K - k^2 \omega_0^2 M + jk\omega_0 C) U_k e^{j\Omega_n k} = \sum_{k=0}^{N-1} F_k e^{j\Omega_n k}. \]  
\( (5-34) \)

Eq. (5-34) is in fact \( N \) complex equations.

\[ (K - k^2 \omega_0^2 M + jk\omega_0 C) U_k = F_k, \quad k = 0, 1, \ldots, N-1. \]  
\( (5-35) \)

The coefficient matrix in (5-35) complex symmetric, so it can be solved using COICCG solver explained in Chapter 2. However if the damping is neglected, the complex equation is decoupled into two systems of equations with real coefficient matrices:

\[ (K - k^2 \omega_0^2 M) U_k^{\text{real}} = F_k^{\text{imag}}, \]  
\( (5-36.a) \)

\[ (K - k^2 \omega_0^2 M) U_k^{\text{real}} = F_k^{\text{imag}}. \]  
\( (5-36.b) \)

Solving (5-35.a) and (5-35.b) is relatively cheaper than solving (5-35). Once \( U \) is found for all \( k \), solution \( u \) can be calculated using (5-31). Fast Fourier transform (FFT) should be used to speed up the computation of (5-30) and (5-31). If the solutions for all \( k \) are sought, it is certainly costlier than time domain method, but we are often looking for a limited number of lowest harmonics. If the number of desired harmonics are much lower than \( N \), the method could be significantly faster.
than the time domain method. In order to calculate FFT of force and also the inverse FFT of the solution, the values at all steps should be available in the memory and this demands a high amount of memory. It is possible to solve this problem by a proper read-write strategy but it may increase the overall computation time. From (5-30) we can verify that

$$U_{N-k} = U_k^*,$$  \hspace{1cm} (5-37)

where * stands for the complex conjugate. Thus, the highest harmonicas are also equally important and they should be included in Eq. (5-31).

5.4.3 Modal Decomposition Method

Consider the following equation which describes the free vibration of the undamped motion equation:

$$M\ddot{u} + Ku = 0.$$  \hspace{1cm} (5-38)

In frequency domain, (5-38) becomes

$$KU = \omega^2 MU,$$  \hspace{1cm} (5-39)

which is a generalized eigenvalue problem. $K$ is a $N\times N$ matrix, thus there are $N$ independent solution to this problem known as eigenvectors associated with $N$ values of $\omega$ which are called eigenvalues satisfying the following equation:

$$Kq_i = \omega_i^2 Mq_i, \hspace{0.5cm} i=0,1,\ldots,N-1.$$  \hspace{1cm} (5-40)

It is clear that the $q_i$ can be multiplied by any non-zero value. The eigenvectors, if scaled properly, have the following properties:
\[ q_j^T M q_i = \begin{cases} 1 & i = j \\ 0 & \text{else} \end{cases}, \quad (5-41.a) \]
\[ q_j^T K q_i = \begin{cases} \alpha_i^2 & i = j \\ 0 & \text{else} \end{cases}. \quad (5-41.b) \]

Solving a generalized eigenvalue shall be explained in Appendix. For the moment, we assume that the eigenvectors and eigenvalues are given. The principal idea of the modal decomposition method is to expand the solution vector of motion equation, at any instances of time, in terms of the eigenvectors. This is perfectly fine because there are \( L \) independent eigenvectors. Thus we have
\[ \mathbf{u}(t) = \sum_{i=0}^{N-1} p_i(t) \mathbf{q}_i. \quad (5-42) \]

By substituting (5-42) into motion equation and multiplying by \( q_j^T \) the following equation is obtained.
\[ \sum_{i=0}^{N-1} \left( q_j^T M q_i \dot{p}_i + q_j^T C q_i \ddot{p}_i + q_j^T K q_i \mathbf{p}_i \right) = q_j^T f. \quad (5-43) \]

In the modal decomposition method, it is necessary to approximate the damping matrix as linear combination of \( M \) and \( K \):
\[ C = \alpha M + \beta K, \quad (5-44) \]
where \( \alpha \) and \( \beta \) are respectively, the mass term and stiffness term damping factors. This is referred to as proportional or Rayleigh damping. With this assumption, we can use (5-41.a) and (5-41.b) to simplify (5-43). The result is the following \( N \) scalar equations:
\[ \ddot{p}_i + (\alpha + \beta \omega_i^2) \dot{p}_i + \omega_i^2 p_i = q_i^T f, \quad i = 1, 2, ..., N - 1. \quad (5-45) \]
The right-hand-side of (5-45) is called the mode participation associated with mode \( j \). Solving \( N \) decoupled scalar equation is very cheap. The general solution of (5-45) is

\[
p_i(t) = A_1 e^{(-\xi + i\sqrt{1-\xi^2})t} + A_2 e^{(-\xi - i\sqrt{1-\xi^2})t},
\]

where the damping ratio \( \xi_i \) is given by the following equation

\[
\xi_i = \frac{1}{2} \left( \frac{\alpha}{\omega_i} + \beta \omega_i \right).
\]

Thus the damping is proportional to \( \alpha \) and \( \beta \) but at higher frequencies, the effect of \( \beta \) on damping increases while the effect of \( \alpha \) becomes smaller.

For the steady-state solution, (5-45) can be solved in frequency domain to obtain \( P_i \). Then \( U_i \) can be found by the following equation:

\[
U_k = Q P_k,
\]

and finally we find the solution of motion equation using (5-31). Although the formulation of the method was based on the complete set of eigenvectors, we often need to consider a limited number of modes and this number is usually much less than the total number because according to (5-46), higher modes damp out rapidly and their contribution to the steady state solution becomes negligible.

As can be seen, if the eigenvectors, \( q_i \), are available, the method has little cost. The computation cost of the method is dominated by the computation of eigenvectors. The cost depends on the number of the required eigenvectors as well as the on matrix size.

The method has a weakness; it is not obvious as how many modes are required to give an acceptable approximation of the solution. In some problems, often when
there is a stiff part on the loading path, the required number is high such that the
time domain method becomes faster and more accurate.

Even for sparse matrices, the eigenvectors are, in general, dense. For small to
moderate size matrices, i.e. up to 1000, the well-known QR method gives the whole
eigenpairs (eigenvalues and eigenvectors). Matrices appearing in the finite element
method usually have over tens of thousands DOF. Fortunately there is a method
known as the subspace iteration method which gives a number of lowest
eigenvalues of large sparse matrices. The subspace iteration method is based on the
power iteration method and also uses the QR algorithm on small matrices in one of
its subroutine. The methods are explained in the following sections.

5.4.4 Model Order Reduction Using Proper Orthogonal Decomposition

We come now to the last solution method of motion equation which is often the
cheapest among the three. The method may serve two purposes, namely order
reduction by projecting high-dimensional data into a lower-dimensional space and
feature extraction by revealing relevant, but unexpected, structure hidden in the data
[55]. The method uses a set of orthogonal basis vectors computed from the limited
number of independent solutions known as snapshots. The orthogonal set fond such
that the energy of the solution in the space snapshots by the set. For this reason, the
method is called proper orthogonal decomposition (POD). The key idea of the POD
is to reduce a large number of interdependent variables to a much smaller number of
uncorrelated variables while retaining as much as possible of the variation in the
original variables. The method is also known as *Principal Component Analysis*, and the *single value decomposition*. The most striking property of the POD is its optimality in the sense that it minimizes the average squared distance between the original signal and its reduced linear representation. Although the method works on nonlinear equations as well, the optimality is only for the linear case. In addition, the cost for nonlinear case is significantly higher [56].

Before describing the method, the projection of the system of equations into a lower dimensional space is explained. Let $u$ be the solution of the following motion equation.

$$ M\ddot{u} + C\dot{u} + Ku = f(t), $$

(5-48)

with $M$, $C$ and $K$ matrices being $N\times N$. The exact solution of the system, in general, can be written in terms of $N$ orthogonal vectors of size $N$.

$$ u(t) = \sum_{i=1}^{N} a_i(t)\psi_i. $$

(5-49)

Fig. 5.2. Schematic diagram showing the transfer matrix $\Psi$ which links the low-dimension vector to the high-dimension.
Now suppose that we would like to approximate the solution in a lower dimensional space, i.e. with a limited number of the independent vectors.

\[ u(t) = \sum_{i=1}^{D} a_i(t) \psi_i, \]  

\[ u(t) = \Psi a(t), \]  

\[ \Psi = [\psi_1, \psi_2, \ldots, \psi_D]. \]  

\[ M \Psi \ddot{a} + C \Psi \dot{a} + K \Psi a = f(t). \]  

The unknown is \( a \) with size \( D \). Eq. (5-53) is a system of \( N \) equations with \( D \) unknowns. Pre-multiplying the equation by \( \Psi^T \):

\[ \Psi^T M \Psi \ddot{a} + \Psi^T C \Psi \dot{a} + \Psi^T K \Psi a = \Psi^T f(t), \]  

\[ M_r = \Psi^T M \Psi, \]  

\[ C_r = \Psi^T C \Psi, \]  

\[ K_r = \Psi^T K \Psi, \]  

\[ f_r(t) = \Psi^T f(t), \]  

\[ M_r \ddot{a} + C_r \dot{a} + K_r a = f_r(t), \]

where \( M_r, C_r, K_r \), and \( f_r \) are the projections of the mass, damping and stiffness matrices and loading, respectively, onto subspace spanned by \( \Psi \). Although the projection matrices could be dense, since \( D \) is usually small, solving reduced equation is very cheap.

The proper orthogonal projection (POD) finds a proper set of basis, i.e. \( \Psi \), to optimally approximate the solution. It works on the basis that the solution of the system contains information about the feature of the system. In the dynamic case, the optimal subspace can be computed from a set of solutions at different time
instances, which are referred to snapshots. In static case a set of independent solution under various loading serves as snapshots. For a set of $D$ basis, $D$ snapshots are required. Let $X$ be an $N \times D$ matrix whose column are the snapshots.

$$X = \begin{bmatrix} u_1 & u_2 & \ldots & u_D \end{bmatrix}. \quad (5-60)$$

Applying the singular value decomposition (SVD),

$$X = V \Sigma W^T \quad (5-61)$$

Where $V$ and $W$ are orthogonal $N \times N$ and $D \times D$ matrices respectively, and $\Sigma$ is an $N \times D$ diagonal matrix of non-negative singular values. The $i^{th}$ row of $W$ represents the entries of the $i^{th}$ column of $X$ projected in the reduced basis formed by the $D$ vectors of $\Sigma$. But

$$\Sigma V = X W, \quad (5-62)$$

therefore, the columns $X W$, after normalization, are the proper orthogonal set.

The cost of singular value decomposition could be prohibitively large due to the size of $X$.

It can be shown that optimal set is the eigenvectors of the sample correlation matrix [56]:

$$R = \frac{1}{D} X^T X, \quad (5-63)$$

because

$$R = \frac{1}{D} (V \Sigma W^T)^T V \Sigma W^T = \frac{1}{D} W \Sigma^2 W. \quad (5-64)$$

The right-hand-side is the eigen-decomposition of $R$, therefore $W$ is the matrix of eigenvectors. Thus

$$\Psi = X W. \quad (5-65)$$
5.5 Numerical Simulations and Comparisons

5.5.1 Vibration of Reactor

Five solution methods described in Chapter 5, i.e. backward Euler, Newmark, frequency domain, modal decomposition methods, and model order reduction using POD method were applied to the vibration analysis of a three-phase reactor core. Fig. 5.3-(a) to (c) describes the model and current waveforms. The model is a three-phase reactor, with 50 turns per each coil. The magnetic and mechanical parameters of the model are typical with the electrical steel. A Three-dimensional magnetic field analysis was carried out and the magnetic force was calculated and stored for the vibration analysis. The number of elements and nodes for this model are 6320 and 8298 respectively.

Motion equation was solved with step size $\Delta t = T/180$ and the assumed damping factors $\alpha = \beta = 10^{-3}$ in time domain and the modal method but in the frequency domain method was applies with zero damping to use (22) and (23). With this time step, only lower harmonics of displacement can be captured. In the modal method, 10 lowest modes and in the frequency domain, 20 lowest harmonics were considered. In time domain methods, to ensure that the solution is not polluted with the transient error, the calculation was done for two periods and the solution of the second period was picked for comparison. For the other two methods, however, one period gives the steady state solution and this is a default advantage. Fig. 5.4-(a) shows the spectra of displacement at a point shown in 5.3-(a). The results show very good agreements between the methods. The discrepancy of the frequency method for
harmonics above 19 is because the higher harmonics were not considered as the frequencies of noise hearable by human ear is below 20 kHz.

Fig. 5.3. (a) Schematic diagram of three-phase reactor, (b) the current waveforms, (c) finite element model of one-quarter region in the structural dynamic analysis. Dimensions are in mm.

Motion equation was solved with $\Delta t=T/180$ and the assumed damping factors $\alpha=\beta=10^{-3}$ in time domain and the modal method but in the frequency domain method was applies with zero damping to use (22) and (23). In the modal method, 10 lowest modes and in the frequency domain, 20 lowest harmonics were considered. In time domain methods, to ensure that the solution is not polluted with the transient error, the calculation was done for two periods and the solution of the second period was picked for comparison. For the other two methods, however, one
period gives the steady state solution and this is a default advantage. Fig. 5.4-(a) shows the spectra of displacement at a point shown in Fig. 5.3-(a). The results shows very good agreements between the methods. The discrepancy of the frequency method for harmonics above 19 is because the higher harmonics were not considered as the frequencies of noise hearable by human ear is below 20 kHz.

![Graph](image)

**Fig. 5.4.** (a) Spectra of $y$-component of the steady state displacement at point P (as shown in Fig. 5.3-(a)) obtained by the Newmark, modal and frequency domain methods, (b) the of CPU times.

The CPU times, shown in Fig. 5.4-(b) and The Computer is Intel(R) Core(TM) i3 CPU @ 3.07GHz with 4GB RAM. It is clear that the POD method is the fastest and very accurate even with 10 snapshots, but the modal method is also fast and the advantage of this method is more than what the numbers show because if the analysis needs to be done under different currents or with different damping factors, the eigenvalues need not to be recalculated, and this make the method very fast. The Euler method turned to be the slowest and also with the highest error. Thus it is
always justified to use the Newmark method if the equation has to be solved in time domain. Fig. 5.5 shows mode shapes and their corresponding frequencies obtained by the subspace iteration methods.

Fig. 5.5. The 10 lowest symmetric modes (symmetries of problem), calculated by the subspace iteration method.

5.5.2 Vibration of a Stiff Model

This problem was solved to investigate the performance of the modal method with limited number of modes applied to larger and stiffer problems. The force is periodic but discontinuous. The model and the force direction is illustrated in Fig. 5.6-(a) and (b). The number of elements and nodes of this model are 54600 and 61152 respectively. Fig. 5.7-(a) shows the displacement of a selected point and Fig. 5.7-(b) compares the computation times the solution methods. It is observed that the
modal method becomes useless here even with 100 modes used. It seems that in this problem, the method requires much higher number of modes are to yield acceptable results, while the frequency domain method with 10 harmonics is slightly faster than the time domain method in the original size of problem. If higher number of harmonics should be considered, the frequency domain method becomes costlier than the Newmark method. The POD, however, is still fast and accurate with the same number of snapshots used in the previous problem.

Fig. 5.6. (a) The stiff model analyzed for comparison. Red arrows show the applied force, (b) the force waveform.
Fig. 5.7. (a) Spectra of $y$-component of the steady state displacement at point P (as shown in Fig. 5.6-(a)) obtained by the Newmark, modal and frequency domain methods, (b) the of CPU times.

5.6 Summary

In this chapter, the dynamic structural analysis was discussed and the semi-discretized finite element equation of motion was derived. The semi-discretized motion equation which is a system of second order ordinary differential equation second order, has to be solved for the solution. There are a number of solution methods to solve the problem. In this thesis five methods were introduced including Euler and Newmark methods in domain, frequency domain method applicable for steady state solution, modal decomposition method, and proper orthogonal method (POD) as a model reduction method. Although all methods could be as accurate as the time domain methods, but due to the size of
matrix, applying some of the methods with their higher accuracy maybe become prohibitively large in term of both CPU time and computer memory. Thus there needs to be a compromised between accuracy and cost with methods other than the time domain with original problem (not the reduced model). The modal decomposition for problems in which the lower modes determine the solution, is fast and accuracy. Accuracy of frequency domain method is predictable as the number of harmonics to include in the analysis can be determined from the problem, while it is always possible to choose enough number of harmonics to obtain a desired accuracy, the computation cost may exceed the Newmark method. The POD method also requires a limited number of snapshots similar to the modes but since the snap, especially with several time steps between them, contain enough information about the solution and thus a relatively low number of snapshots is required. This is the reason that POD is accurate and fasters in both problems. Yet, the safest method to be used is the time domain method, in particular the Newmark method. Thus In problems were prediction solution is not easy; it is safest to use the Newmark method.
Chapter 6 Vibration Analysis of AC Permanent Magnet Motor

6.1 Introduction

Vibration analysis of electric motors has attracted considerable research attention recently as designers are facing tighter criteria of low acoustic noise emission. Due to higher torque density and lower loss, IPM motors are finding increased application in home appliances as well as electric vehicles and design of low-noise motors requires accurate calculation of vibration and in-depth understanding of vibration characteristics of the motor. A particular challenge in the vibration analysis of IPM motors is that the input voltage usually has sinusoidal pulse width modulation (SPWM) waveform with switching frequencies up to hundred times as high as base frequency. The switching frequency harmonics can affect the vibration of the motor through magnetic and MS forces. Calculation of the coil currents with such a high switching frequency requires very small time step size in conjunction with very small change in the rotor position. Some researchers have evaded the difficulty of current calculation by using measured currents in the analysis, but for a motor in the design stage, this option is not available. In this paper, a 2-D magnetic field analysis of an 8-pole 36-slot IPM motor with SPWM input voltage is carried out and the coil currents are calculated. Then the effect of the MS on the vibration characteristics of the motor is investigated by a 3-D finite element structural dynamic analysis.
6.2 Two-Dimensional Magnetic Field Analysis Coupled with Circuit Equation

Fig. 6.1 shows the cutaway diagram of the 8-pole 36-slot IPM motor. Neglecting the edge effect, a 2-D magnetic field analysis coupled with circuit equations is carried out to obtain the coil currents. Exploiting the rotational symmetries of the magnetic model, only a region needs to be analyzed. The finite element model of the magnetic domain of analysis is shown in Fig. 6.2 and Fig. 6.3 shows the schematic circuit diagram of inverter and motor connections.
Fig. 6.2. Finite element model of magnetic domain of analysis. The rotor is rotated with discrete rotation steps of 0.05 degree.

The analysis condition is given in TABLE 6.1, and Fig. 6.4-(a) to (d) show the SPWM waveform generation. The frequency of reference voltages $V_{ra}$, $V_{rb}$ and $V_{rb}$ is 100 Hz with 120 electrical degree phase differences. The frequency of the triangular waveform in the figure is shown one tenth of the actual value for better illustration of the SPWM waveforms.

Fig. 6.3. Circuit diagram of motor connected and inverter connections.
The field circuit equations can be written in the following forms:

\[
\nabla \times (\nabla \times A) + \frac{Nh_i}{S} = \left( \frac{\partial M_x}{\partial x} - \frac{\partial M_y}{\partial y} \right),
\]

\[
L \frac{di_a}{dt} + Ri_a - \frac{d\phi_u}{dt} + V_m = V_a,
\]

\[
L \frac{di_b}{dt} + Ri_b - \frac{d\phi_v}{dt} + V_m = V_b,
\]

\[
L \frac{di_c}{dt} + Ri_c - \frac{d\phi_w}{dt} + V_m = V_c,
\]

\[
R_m \gg 1 \rightarrow i_a + i_b + i_c = \frac{V_m}{R_m} = 0,
\]

\[
\phi_{ij} = \sum_j \left( \frac{hN_{ij}^l}{S_{ij}} \int_{s_{ij}} Ads \right) - \sum_j \left( \frac{hN_{ij}^l}{S_{ij}} \int_{s_{ij}} Ads \right),
\]

and similar equations for \( \phi_v \) and \( \phi_w \). In above equations, \( i_k \) is the current of either phase or depending on the point, it could be outward or inward, \( \phi \) is the flux interlinking, \( M_x \) and \( M_y \) are the components of the PM magnetization, \( i_a, i_b, \) and \( i_c \) are the phase currents, \( L \) is the leakage inductance, \( R \) is the winding resistance, \( h \) is the stator length in the third dimension, \( A \) is the z-component of the vector potential.

Subscripts + and – for the coils indicate that the coil currents in flowing are out of or in to the plane respectively. The above equations are discretized by triangular finite elements and are solved using the Newton-Raphson method as described in Subsection 2.5.3. The material magnetization data is shown in Fig. 6.5. It is noteworthy that the effect of stress on the flux density distribution was not considered in this study, therefore, an uncoupled magnetic field analysis was carried out.
The mesh was constructed such that the rotor can be rotated by discrete rotation steps of 0.05 mechanical degree without deforming elements. The switching frequency is adjustable between 1 to 15 kHz but in this paper the calculations are carried out for the frequency of 7.2 kHz and with the time step size $\Delta t=50/9$ microseconds, equivalent of 0.05 mechanical degree which requires 1800 steps of rotation is in one electric cycle. The mesh consists of 20558 first order triangular elements out of which more than three quarter of the elements belong to the air gap region. Such a high density mesh in the air gap allows the rotation of the rotor by 0.05 degree without deforming elements or using non-conforming techniques.

In solving the coupled field-circuit equations we mainly followed the lines given in [48-50]. For a given torque and torque angle, the currents can be controlled by adjusting the level of the reference voltages $V_{ra}$, $V_{rb}$ and $V_{rb}$ shown in Fig. 6.4-(a), while rotating the rotor with the synchronous speed. Fig. 6.6-(a) shows the calculated steady state currents under conditions specified in TABLE 6.1, and the current harmonics from 0 to 20 kHz is shown in Fig. 6.6-(b). Note that the spectrum
is based on the exponential Fourier expansion, therefore half the magnitude appears in the spectrum of positive frequency range. The magnetic flux distribution in the stator corresponding with rotor position of 45 degree is shown in Fig. 6.7-(a) and the instantaneous torque is shown in Fig. 6.7-(b).

**TABLE 6.1**

Motor specifications for magnetic field analysis

<table>
<thead>
<tr>
<th>Specification</th>
<th>Unit</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poles</td>
<td>-</td>
<td>8</td>
</tr>
<tr>
<td>Slots</td>
<td>-</td>
<td>36</td>
</tr>
<tr>
<td>Turn per slot</td>
<td>-</td>
<td>62</td>
</tr>
<tr>
<td>Magnetization of permanent</td>
<td>T</td>
<td>1.21</td>
</tr>
<tr>
<td>Air gap</td>
<td>mm</td>
<td>0.6</td>
</tr>
<tr>
<td>Core material</td>
<td>-</td>
<td>50A400</td>
</tr>
<tr>
<td>Housing and shaft material</td>
<td>-</td>
<td>S45C</td>
</tr>
<tr>
<td>Current</td>
<td>A</td>
<td>6</td>
</tr>
<tr>
<td>Nominal torque</td>
<td>N.m</td>
<td>9.6</td>
</tr>
<tr>
<td>Torque angle</td>
<td>degree</td>
<td>0</td>
</tr>
<tr>
<td>Frequency</td>
<td>Hz</td>
<td>100</td>
</tr>
<tr>
<td>SPWM carrier frequency</td>
<td>Hz</td>
<td>7200</td>
</tr>
<tr>
<td>Winding resistance</td>
<td>Ω</td>
<td>0.6</td>
</tr>
<tr>
<td>Leakage inductance</td>
<td>mH</td>
<td>8 (assumed)</td>
</tr>
<tr>
<td>Stator length</td>
<td>mm</td>
<td>50</td>
</tr>
<tr>
<td>Inverter DC voltage</td>
<td>V</td>
<td>288</td>
</tr>
</tbody>
</table>
Fig. 6.5. B-H data 50A400 (core) and S45C (shaft and housing).

Fig. 6.6. (a) Calculated currents under conditions specified in TABLE 6.1, (b) current harmonics in the range of 0 to 20 kHz.
Fig. 6.7. (a) Magnetic flux density in the stator corresponding to rotor position of 45 degree, (b) instantaneous torque.

As shall be explained in the next section, the rotor vibration becomes irrelevant to our study. The magnetic and MS forces affecting the housing vibration are those acting on the stator. Since the stator is laminated, it is best to calculate the forces in 2-D and then map it onto the 3-D model considering the lamination space factor. The 2-D magnetic nodal force is calculated using the Maxwell stress tensor by the following equations [27]:

\[ F_i^m = -\sum_{s_i} \left[ T_e^m \right]_{ij} \nabla N_j \, ds, \tag{6-7} \]

\[
[T^m] = \begin{bmatrix} B_x H_x - p^m & B_x H_y \\ B_y H_x & B_y H_y - p^m \end{bmatrix}, \tag{6-8} \]

\[ p^m = B \cdot H - \int_0^H \nabla' \cdot dB', \tag{6-9} \]

where \( T^m \) is the magnetic stress tensor, \( B \) and \( H \) are the magnetic flux density and magnetic field intensity respectively, and \( p^m \) is the magnetic pressure [27].
Magnetostriction in steel sheets used in manufacturing stator cores also show strong anisotropy even in magnetically isotropic materials. The MS in transverse direction could be several times larger than in rolling direction [67]. This phenomenon can be explained as the effect of stress; rolling the sheet results in a significant residual tensile stress in rolling direction which reduces the MS strain. To reduce this anisotropy, cores are made by transposing every other laminate, so the effective MS of the core can be approximated by the average MS of the rolling and transverse directions. Fig. 6.8 shows typical anisotropic $\lambda$ curves for rolling and transverse directions and their average in the homogenized anisotropy. The assumed $\lambda$ curves for three levels if stress, for use in this study is shown in Fig. 6.9.

Fig. 6.8. Magnetostriction anisotropy and the average magnetostriction of laminates.
Fig. 6.9. Assumed MS curve for the core under stress levels of 0, 50 MPa and 100 MPa.

Since the flux distribution in the core is not uniform and also restrictions and boundary conditions may prevent free change of dimensions, the deformation of the core requires a structural analysis with equivalent MS nodal forces which is given by the following equations [33]:

\[ F_{i}^{ms} = - \sum_{e} \left( T_{e}^{ms} \right)^{T} N_{i} ds \]  \hspace{1cm} (6-10)  

\[ \left[ T_{ms}^{e} \right] = \frac{-E \lambda}{2(1+\nu_{p}) B^{2}} \begin{bmatrix} 3B_{x}^{2}-tB_{y}^{2} & 3B_{x}B_{y} \\ 3B_{x}B_{y} & 3B_{y}^{2}-tB_{x}^{2} \end{bmatrix}, \] \hspace{1cm} (6-11)

where \( T_{ms}, \lambda, E \) and \( \nu_{p} \) are the MS stress tensor, MS parameter, Young’s modulus, and Poisson’s ratio, respectively. If the 2-D model is the plane stress approximation of the actual 3-D model

\[ t = \frac{1 - 2\nu_{p}}{1 - \nu_{p}}, \] \hspace{1cm} (6-12)

and for the plane strain approximation \( t = 2 \). In thin plates like electrical steel sheets as in stator core, the stress along the thickness is negligible, so the plane stress approximation is used.
The magnetic and total forces at one instance of time are shown Fig. 6.10-(a) and Fig. 6.10-(b), respectively. The significant magnetic nodal forces are mainly found on the stator teeth and they depend on the magnetic flux density in the air regions whereas those MS forces affecting the radial vibration are mainly found at rear of the slots.

![Fig. 6.10. (a) Magnetic force, (b) total force (sum of magnetic and MS forces).](image)

### 6.3 Shrink-Fit Stress

With the dimensions given in Figs. 6.2, and 6.11, radial interference of $\delta=0.048$ mm was used in the thermal force calculation according to manufacturer. Only one segment of the stator-housing structure was analyzed due to the rotational symmetry, and a fine mesh was constructed to take account of the laminated structure. Model dimensions and mechanical parameters are listed in TABLE 6.2. Figs. 6.12-(a) and
(b) show the circumferential and radial stresses, respectively. The stress level is high enough to affect the loss and magnetostriction significantly.

**TABLE 6.2**

<table>
<thead>
<tr>
<th>Specification</th>
<th>Uni</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>laminate Young’s modulus</td>
<td>Pa</td>
<td>2×10¹¹</td>
</tr>
<tr>
<td>insulator Young’s modulus</td>
<td>Pa</td>
<td>1×10¹⁰</td>
</tr>
<tr>
<td>housing Young’s modulus</td>
<td>Pa</td>
<td>2×10¹¹</td>
</tr>
<tr>
<td>laminate thickness</td>
<td>mm</td>
<td>0.5</td>
</tr>
<tr>
<td>insulator thickness</td>
<td>mm</td>
<td>0.02</td>
</tr>
<tr>
<td>all Poisson’s ratios</td>
<td>-</td>
<td>0.3</td>
</tr>
</tbody>
</table>

**Fig. 6.11.** A segment of stator-housing structure.

**Fig. 6.12.** (a) Thermal force at x-y plane, (b) and (c), circumferential and radial shrink-fit stresses, respectively.
6.4 Core Loss of Stator

The loss calculation method used in this study was discussed in Section 2.13. In this section, the method is applied to loss calculation of the motor. The loss characteristics of 50A400 (equivalent of M43) is shown in Fig. 6.13-(a) [62] and the hysteresis coefficient of Bertotti model is shown in Fig. 6.14-(a) [62]. Fig. 6.13-(b) shows loss vs. stress for various grade of steel at 1.0 T, 400 Hz [63]. The thinness of 50A400 is 0.50 mm while the thickness samples in Fig. 6.13-(b) is 0.20 mm, but assuming that the hysteresis part of the specific loss does not depend on the thickness, we can use the data of Fig. 6.13-(b) to obtain coefficient of stress-dependency $C_\sigma$ in the following equation:

$$w(B, f, \sigma) = w(B, f, 0) + k_h C_\sigma B^\sigma f$$ (6-13)

Since $B=1.0$ T, coefficient $\alpha$ become irrelevant. Assuming 2.5% Si for the material, $C_\sigma$ is estimated as described in Fig. 6.14-(b). It shall be shown that the stress due to magnetic and magnetostriction forces are too low to consider in the loss calculation, therefore the shrink-fit stress presented in Section 6.3 was used together with the loss data and parameters and the loss distribution and overall loss of the stator was calculated. The results are shown in Figs. 6.15-(a) and (b). It can be seen that the loss has risen significantly due to stress. The loss of the 3\textsuperscript{rd} and 5\textsuperscript{th} harmonics of the flux density are less affected by the stress because the flux in the stressed region, back-iron, is mainly sinusoidal and alternating.
Fig. 6.13. (a) Core loss characteristics of electrical steel of grade 50A400 under zero stress, (b) loss v.s. stress at 1.0 T, 400 Hz for various grades of steel labeled by silicon content.

<table>
<thead>
<tr>
<th>Induction [T]</th>
<th>Frequency [Hz]</th>
<th>( k_b ) [W/Hz/T^3]</th>
<th>( \alpha ) [-]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B &lt; 0.7 )</td>
<td>20</td>
<td>0.0066</td>
<td>1.2184</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>0.0053</td>
<td>0.9518</td>
</tr>
<tr>
<td></td>
<td>120</td>
<td>0.0101</td>
<td>1.7305</td>
</tr>
<tr>
<td></td>
<td>300</td>
<td>0.0131</td>
<td>2.0347</td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>0.0135</td>
<td>2.1005</td>
</tr>
<tr>
<td>( 0.7 &lt; B &lt; 1.4 )</td>
<td>20</td>
<td>0.0099</td>
<td>1.8648</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>0.0099</td>
<td>1.8906</td>
</tr>
<tr>
<td></td>
<td>120</td>
<td>0.0102</td>
<td>1.9086</td>
</tr>
<tr>
<td></td>
<td>300</td>
<td>0.0110</td>
<td>1.9821</td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>0.0113</td>
<td>2.0656</td>
</tr>
<tr>
<td>( B &gt; 1.4 )</td>
<td>20</td>
<td>0.0161</td>
<td>1.0681</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>0.0105</td>
<td>1.9251</td>
</tr>
<tr>
<td></td>
<td>120</td>
<td>0.0110</td>
<td>1.8642</td>
</tr>
<tr>
<td></td>
<td>300</td>
<td>0.0095</td>
<td>2.1767</td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>0.0071</td>
<td>2.9065</td>
</tr>
</tbody>
</table>

Fig. 6.14. (a) Hysteresis coefficient of Bertotti model for 50A400, (b) estimated stress-dependency coefficient for steel of grade 2.5%.

\[
C_\sigma = -k_2(\sigma - \sigma_1) + C_1
\]

\[
k_1 = 0.088 \text{ MPa}^{-1}
\]

\[
k_2 = 0.015 \text{ MPa}^{-1}
\]

\[
\sigma_1 = 20 \text{ MPa}
\]
Stress considered

14.1 W/kg

Stress neglected

14.1 W/kg

(a) Stress neglected

(b) Stress considered

Due to stress
Without stress

Overall loss

10.4
13.2
0.2
0.7
0.5
0.1

Flux harmonic order

Stator core loss (W)

10.0
5.0
0.0

(c) Overall loss

Fig. 6.15. Distribution of specific loss in the stator (a) neglecting, (b) considering the shrink-fit stress, (c) overall stator loss for first three odd harmonics.

6.5 Homogenization of Laminated Structure of Stator for Structural Analysis

Stator has a laminate structure comprised of layers of electrical steel and very thin layers of insulator which is often much softer than iron and with a mass density lower than that of iron. Therefore it affects the stiffness, mass and also damping of the structure. It is impractical to construct a finite element model which takes account of the laminates and the insulators. Instead a homogenized model is assumed with material parameters reflecting the effect of inhomogeneity. Such tricks are known as homogenization techniques.

The homogenized model of the stator is no more isotropic. It becomes orthotropic, i.e. the material properties are, in general, different along the three
perpendicular direction. The orthogonal parameters are calculated using the rule of mixtures [58]:

\[ E_x = E_y = f_{steel} \times E_{steel} + f_{ins} \times E_{ins} \]  
\[ (6-14) \]

\[ v_{xy} = f_{steel} \times v_{steel} + f_{ins} \times v_{ins} \]  
\[ (6-15) \]

\[ \frac{1}{E_z} = \frac{f_{steel} \times E_{steel} + f_{ins} \times E_{ins}}{E_{steel} + E_{ins}} \]  
\[ (6-16) \]

\[ \frac{1}{v_{xz}} = \frac{f_{steel} \times v_{steel} + f_{ins} \times v_{ins}}{v_{steel} + v_{ins}} \]  
\[ (6-17) \]

\[ G_{xy} = f_{steel} \times G_{steel} + f_{ins} \times G_{ins} \]  
\[ (6-18) \]

\[ \frac{1}{G_{xz}} = \frac{f_{steel} \times G_{steel} + f_{ins} \times G_{ins}}{G_{steel} + G_{ins}} \]  
\[ (6-19) \]

where \( E, \nu, \) and \( G \) are Young’s modulus, Poisson’s ratio, and Shear modulus, and \( f_{steel} \) and \( f_{ins} \) are the so-called filling factors of steel and insulator, respectively. The filling factors are defined according to the following equations:

\[ f_{steel} = \frac{\text{thickness of steel}}{\text{thickness of steel} + \text{thickness of insulator}} \]

\[ f_{ins} = \frac{\text{thickness of insulator}}{\text{thickness of steel} + \text{thickness of insulator}} \]

It is obvious that \( f_{steel} + f_{ins} = 1 \).

As for the shear modulus of isotropic materials we have

\[ G_{steel} = \frac{E_{steel}}{2(1+\nu_{steel})} \]  
\[ (6-20) \]

and similar equation for insulator.
6.6 Three-Dimensional Mechanical Modelling of the Motor

A 3-D model of the motor was constructed by piling up on the 2-D mesh and adding front and rear covers, bearing housing, bolts and the bases. Fig. 6.16 shows half region of the mechanical FEM model. Neglecting small differences between the front and the rear, we can use the place symmetry and this reduces the domain of analysis into half-region. Mechanical parameters are given in TABLE 6.3.

Fig. 6.16. Half-region finite element mesh with wedge elements.
<table>
<thead>
<tr>
<th>Specification</th>
<th>Unit</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>laminate Young’s modulus</td>
<td>Pa</td>
<td>2×10^{11}</td>
</tr>
<tr>
<td>insulator Young’s modulus</td>
<td>Pa</td>
<td>1.0×10^{10}</td>
</tr>
<tr>
<td>laminate and insulator Poisson’s ratio</td>
<td>-</td>
<td>0.3</td>
</tr>
<tr>
<td>laminate mass density</td>
<td>kg/m^3</td>
<td>7800</td>
</tr>
<tr>
<td>insulator mass density</td>
<td>kg/m^3</td>
<td>1500</td>
</tr>
<tr>
<td>housing Young’s modulus</td>
<td>Pa</td>
<td>2×10^{11}</td>
</tr>
<tr>
<td>housing Poisson’s modulus</td>
<td>-</td>
<td>0.3</td>
</tr>
<tr>
<td>laminate thickness</td>
<td>mm</td>
<td>0.5</td>
</tr>
<tr>
<td>insulator thickness</td>
<td>mm</td>
<td>0.02</td>
</tr>
</tbody>
</table>

Since the 3-D model was constructed by piling up on the 2-D model, the forces calculated in 2-D model can be simply transferred to the 3-D model by multiplying the 2-D forces, which are in fact in units of N/m, by the thickness of the element layers and half the product goes to the nodes on each side of the layer the following equation:

\[
f_{3-D} = \frac{1}{2} (\Delta_{upper} + \Delta_{lower}) f_{2-D}
\]  

(6-21)

\(\Delta_{upper}\): thickness of upper iron element layer

\(\Delta_{lower}\): thickness of lower iron element layer

If not piled up, the force in 3-D model should be interpolated to obtain the force at 3-D nodes. Fig. 6.17 shows the force distributions in 3-D model.
6.7 Vibration Analysis of Motor

Recall the motion equation from Chapter 5:

\[ M\ddot{u} + C\dot{u} + Ku = f, \]

(6-22)

where \( M, C \), and \( K \) are the mass, damping and stiffness matrices, respectively, \( u \) is the displacement, and \( f \) is the total force, i.e. \( f = f^m + f^{ms} \). The proportional damping is used for approximation of \( C \) [46]:

\[
[C] = \alpha[M] + \beta[K],
\]

(6-23)

where \( \alpha \) and \( \beta \) are the mass and stiffness proportional damping factors, respectively.

The damping ratio \( \zeta_k \) associated with mode \( k \) is linked to the damping factors according to the following equation [46]:

\[
\zeta_k = \frac{1}{2} \left( \frac{\alpha}{\omega_k} + \beta \omega_k \right),
\]

(6-24)
where $\omega_k$ is the angular frequency of mode $k$. The parameters can be measured by exciting two lowest natural frequencies of the structure and measuring the decay rate. However we carried out the investigation with assumed values. Since the modes of motor are in the range of thousands of Hertz and $\alpha$ and $\beta$ are usually much smaller than unity, $\alpha$ is of no importance whereas the damping at higher frequencies strongly depends on $\beta$. In this paper we assume $\alpha=0$ and $\beta=10^{-5}$.

Vibration analysis is carried out once with magnetic force only and once with the total force; the sum of magnetic and magnetostriction neglecting stress and once considering stress. For comparison of computation costs, only the Newmark and POD methods are compared.

One electric cycle is 1800 time steps and for solving in time domain method, both original system and reduced order system (POD), the motion equation was solved for two cycles to reach the steady state solution and the second period solutions were included in the analyses and the comparisons.
6.8 Numerical Results and Discussions

6.8.1 Comparison Between Solution Methods

First a comparison between the solution methods in this specific problem is made. The frequency domain becomes costlier than the time domain for the vibration analysis because 200 harmonics should be considered. The comparison is made between the Newmark and POD methods with 100 snapshots. The time step is 0.01/1800 seconds, damping factors are $\alpha=0$ and $\beta=10^{-5}$, and the loading is the magnetic force. The results are shown in Fig. 6.18. It is obvious that enormous speed-up together with a good accuracy is gained by applying POD. Therefore the result of calculation is carried out using POD method.

![Figure 6.18.](image)

(a) Comparison of velocity spectra
(b) Comparison of computation times

Fig. 6.18. (a) Spectra of radial velocities at a point P, (b) comparison of computation times. The Computer is Intel(R) Core(TM) i3 CPU @ 3.07GHz with 4GB RAM.

6.8.2 Comparison between MS-Considered and MS-Neglected Analyses
The radial displacements of a point on the housing surface are compared in Fig. 6.19-(a), and a graphical comparison of enlarged deformation of is shown in Fig. 6.19-(b). Although the effect of MS, when stress considered, on the radial displacement is significant, the difference is nearly time-invariant. Since the normal velocity of surface contributes to acoustic noise, the time-invariant term becomes irrelevant.

Fig. 6.19. Comparison of displacements due to magnetic and total forces at a point P, (b) an instant shot of enlarged (110000 times) deformations of the frame.

To investigate the effect of MS and stress on the velocities more accurately, the spectra of radial and circumferential velocities are shown in Fig. 6.20-(a) and (b), respectively. Relatively large harmonics appear at inverter switching frequency and its second multiple, as expected. The circumferential velocity is one order of magnitude smaller the radial velocity. Moreover, only normal velocity (here radial) contributes to noise, therefore, only radial velocity is discussed. It is revealed that the MS under shrink-fit stress reduces higher harmonic of significant amplitudes
and since higher harmonics, as long as in the acoustic range, generate more pressure and hence more noise, it can be concluded that the MS under shrink-fit stress slightly reduces the acoustic noise of motor by partially cancelling the noise due to magnetic force. It should be noted that these results are based on the assumed damping factor of $10^{-5}$. For higher damping factors, the harmonics of switching frequency are damped, therefore the effect of MS becomes less pronounced. Fig. 6.20-(c) shows the A-weighting standard of human ear sensitivity to noise harmonics. The first harmonic of switching frequency then is heard almost without weakening, but the second harmonic is weighted by about -5 dB.

![Fig. 6.20. Comparison of radial and circumferential velocity spectra.](image_url)
6.8.3 Stress Distribution in Stator

The shrink-fit stress was shown in Section 6.3. Similarly, the stresses due to magnetic and MS forces were be calculated from the solution of motion equation. The stress is time-dependent but instances of the stresses are shown in Figs. 6.21-(c) to (f) for comparison with the shrink-fit stress in Figs 6.21-(a) to (b). Different symmetries were used due to different loading symmetries. It is verified that the stresses due to magnetic and MS forces are negligible compared with the shrink-fit stress, our loss and vibration calculation based o shrink-fit stress is valid.

Fig. 6.21. Comparison of stresses (a) and (b) shrink-fit stress, (c) and (d) due to magnetic force, (e) and (f) due to MS force considering the shrink-fit stress
6.9 Summary

We carried out a 2-D magnetic field analysis of IPM motor for the calculation of loss and vibration. The loss was calculated using proposed method of taking account of the shrink-fit stress. It was observed that the stress gives rise to the stator loss by nearly 78%.

Next, 3-D vibration analyses were carried for three cases of: MS and stress neglected, MS considered and stress neglected, and both MS and stress considered. With assumed typical MS parameter for the stator and assume damping for the motor, it was observed that the MS, especially under the effect of shrink-fit stress, slightly reduces the vibration of housing surface, hence the acoustic noise. It was also observed that the current harmonics can induce significant to large vibration harmonics for the assumed damping of the motor. According to the results, by applying more shrink-fit stress or by using materials with higher MS parameter, the acoustic noise can be reduces but the loss would increases. For situation where the loss is less important that the noise, this can be used to reduce the noise emission.

A comparison was also made between two solution methods of motion equations applied to the motor analysis; the Newmark method and the proper orthogonal decomposition (POD) method. It was observed that by adopting POD, enormous speed-up can be gained with good accuracy.
Chapter 7  Conclusions and Recommendations

In this thesis, the interdisciplinary topic of magneto-mechanical analysis was discussed thoroughly in the context of electrical machines and magnetic devices. The study involves not only the magnetic field analysis and loss calculation but also structural static and dynamic analysis. It goes as far as discussing solution methods of eigenvalue problem as it become relevant to vibration characteristics of magneto mechanical devices.

One aim of the study is the investigation of the effect of stress on the performance of electrical machines and magnetic devices. Considering the effect of magnetically induced stress on the magnetic flux distribution in the device is challenging because the stress can alter the magnetic field and this makes the problem two way coupled. However, in most magnetic devices the level of stress is relatively low. In addition, in such devices there often exists air gaps in the magnetic path which determines the magnetic flux density. Thus a small variation of magnetic permeability has no practical effect on the magnetic flux distribution. Two methods were proposed to take account of stress in the coupled magnetic field analysis. In both methods, instead of measured magnetization data under stress, it requires with magnetostriction data under stress as well as the stress-free magnetization data. Using the balance of mechanical and magnetic energies, the method calculates the magnetic permeability of the material as a function of the magnetostriction and stress as well as the magnetic flux density.
Shrink-fitting of motor induces high levels of stress in the stator which affects the iron loss and vibration. The stress may also affect the flux distribution by altering the magnetic property of the stator core, but this was not investigated in this study.

The vibration analysis in electrical devices was discussed in details as one of the aims of the study is the calculation of vibration in devices that are powered through inverter power supplies. In such cases, the voltage source and thus the magnetic flux density contains high harmonics which could be reflected in the vibration of the device. Taking account of such harmonics requires solving magnetic and structural dynamic analysis with very small time steps. This could add heavy computational cost. In this thesis, the possibility of reducing the cost was studied. A number of solution methods of motion equation, which is often linear in the vibration analysis, were applied to typical problems and comparison were made between them regarding the accuracy and cost. Advantages and drawbacks of each method were observed in the comparison. One is that the performance of modal decomposition method with limited number of modes, which is a popular method on structural dynamic analysis, is highly problem-dependent. The issue is that it may not be obvious before-hand whether the method gives good approximation of the solution. A relatively newer method known as the proper orthogonal decomposition proved to be superior to the modal method in terms of both accuracy and cost. The frequency domain method, with limited number of harmonics, is also valid and its accuracy is predictable but the cost may exceed that of time domain method. The time domain method, in particular the Newmark method, is the most accurate and thus safe to use in problems where the feature of solution is not known.
Core loss calculation of IPM motor taking account of stress was another aim of this study. We carried out a 2-D magnetic field and then the loss was calculated using proposed method of taking account of the stress. It was observed that the shrink-fit stress gives rise to the stator loss by nearly 78%.

Investigating the effect of magnetostriction property of the material on the mechanical response of the device, namely the vibration, was another key topic in this thesis. The study carried out vibration analysis of a permanent magnet synchronous motor driven by an SPWM inverter power supply. Calculations were done for three cases; MS and stress neglected, MS considered and stress neglected, and both MS and stress considered. It was observed that the magnetostriction, when under shrink-fit stress, slightly reduces the vibration by partially cancelling the vibration due to magnetic force.

It is important to note that several assumptions made in these analysis. The assumption, which may strongly affect the results and hence the conclusions, are listed below:

- Effect of stress on the flux distribution was neglected.
- Eddy currents were neglected.
- Not actual but typical magnetostriction parameters was used.
- Effect of frequency on magnetostriction parameter was neglected.
- Only hysteresis part of core loss was considered stress-dependent.
- Assumed damping factor of $10^{-5}$ used for the motor structure. Higher damping factors results in lower vibration.
Appendix

A.1 Solution Methods of Generalized Eigenvalue Problem

Recall the undamped equation of motion:

\[ M\ddot{u} + Ku = F(t) \]  \hspace{1cm} (A-1)

If the force is removed we have

\[ M\ddot{u} + Ku = 0 \]  \hspace{1cm} (A-2)

In frequency domain we have

\[ KU = \omega^2 MU \]  \hspace{1cm} (A-3)

where \( U = U(j\omega) \) is the Fourier transform of \( u(t) \). Eq. (A-3) has the form of the generalized eigenvalue problem:

\[ Kx = \lambda Mx \]  \hspace{1cm} (A-4)

where \( x = U \), and \( \lambda = \omega^2 \)

It can be proven that if \( K \) and \( M \) are symmetric, the eigenvalues are real. Furthermore, if they are positive definite, the eigenvalues are positive. When \( M \) is invertible we can multiply both side by its inverse and arrive at the standard eigenvalue problem:

\[ Ax = \lambda x \]  \hspace{1cm} (A-5)

where \( A = M^{-1}K \).

It can be shown that the values of \( \lambda \) satisfying (3-3) are the roots of the following equation:
\[
\det(A - \lambda I) = 0, \quad (A-6)
\]
where \(\det(.)\) stands for the determinant of matrix and \(I\) is the Identity matrix. Eq. \(A-6\) has \(n\) roots where \(n\) is the size of \(A\).

Except for a very small \(n\), finding roots of the polynomial is impractical and the computation of eigenvectors after finding the eigenvalues, is also exhausting. There are methods for the calculation of the eigenpairs of a matrix without the need to solve \((A-6)\). Here we introduce three methods which are relevant to our study:

1. Power iteration method
2. QR algorithm
3. Subspace iteration method

A.1.1 Power iteration method

The power iteration method gives the finds the largest eigenvalue. The standard eigenvalue problem has the following form:

\[
Ax = \lambda x, \quad (A-7)
\]
where \(A\) is an \(n \times n\) square matrix. The matrix has \(n\) independent eigenvectors \(q_i\) (\(i=1,2,\ldots,n\)) corresponding to eigenvalues \(\lambda_i\) (\(i=1,2,\ldots,n\)) such that \(|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|\).

spanning the \(n\)-dimensional space, thus a random vector \(x\) can be written as a linear combination of these eigenvalues:

\[
x = \sum_{i=1}^{n} c_i q_i. \quad (A-8)
\]

By multiplying \(A\) by \(x\) we have the vector \(y_1\) as follows
\[ y_1 = \sum_{i=1}^{n} c_i A q_i = \sum_{i=1}^{n} c_i \lambda_i q_i \] \hspace{1cm} (A-9)

If we keep pre-multiplying \( x \) by \( A \), the product vector after \( k \)-th multiplication becomes

\[ y_k = A^k x = \sum_{i=1}^{n} c_i \lambda_i^k q_i . \] \hspace{1cm} (A-10)

As \( k \) becomes larger, the first term dominates the summation so

\[ \lim_{k \to \infty} y_k = \lambda_1^k q_1 , \] \hspace{1cm} (A-11)

thus

\[ \lambda_1 = \lim_{k \to \infty} \frac{|y_k|}{|y_{k-1}|} \text{ , and } q_1 = \lim_{k \to \infty} \frac{y_k}{|y_k|} . \] \hspace{1cm} (A-12)

Therefore the power iteration method gives the largest eigenvalue and its corresponding eigenvector. It is noteworthy that if the starting vector happens to be orthogonal to \( q_1 \), then \( c_1 = 0 \) and the power method gives the second largest eigenvalue.

The method can be slightly modified to give the smallest eigenvalue; by multiplying both sides of the standard eigenvalue problem by \( A^{-1} \), the following equation is obtained:
\[ A^{-1}x = \frac{1}{\lambda}x, \quad (A-13) \]

which is also in the form of the standard eigenvalue problem. The eigenvalues of (A-13) are the inverse of the eigenvalues of the original problem, thus the largest eigenvalue of (A-13) is the smallest eigenvalue of (A-7). In practice there is no need to inverse the matrix because, instead of calculating \( y = A^{-1}x \), we can solve \( Ay = x \).

The method can also be applied to the generalized eigenvalue problem which has the following form:

\[ Kx = \lambda Mx. \quad (A-14) \]

To this end, equation can be written in the following form:

\[ K^{-1}Mx = \frac{1}{\lambda}x. \quad (A-15) \]

To apply the power iteration method we need to calculate \( y = K^{-1}Mx \) which is equivalent of solving the following equation:

\[ Ky = Mx \quad (A-16) \]

The QR algorithm and the subspace iteration method explained in the following sections are based on the power iteration method. QR algorithm calculates the whole and the subspace iteration calculates desired number of smallest eigenvalues.

A.1.2 QR method

The method is based on the QR decomposition of the matrix, i.e., writing the matrix as a product of an orthogonal matrix \( Q \) (\( Q^TQ=I \)) and an upper triangular matrix \( R \), gives the complete eigenvectors and well as eigenvalue. The method
works by decomposing the matrix into $Q$ and $R$ and multiplying them in the reverse order and iterating until the product converge to a triangular matrix, the Schur form of the matrix. The algorithm is given below:

Let $A$ be the matrix whose eigenpairs is to be calculated.

$A_1 = A$

$P = I$

Repeat for $k=1, 2, \ldots$

1. $A_k = Q_k R_k$ (performing QR decomposition)

2. $A_{k+1} = R_k Q_k$

3. $P = PQ_k$

if convergence achieved then end.

After convergence, the diagonal entries of $A_{k+1}$ are the eigenvalue of the original matrix $A$ because

\[ A_{k+1} = R_k Q_k = Q_k^T R_k Q_k = Q_k^T A_k Q_k \]  \hspace{1cm} (A-17)

Thus all $A_k$ are similar matrices. In general $B = P^{-1}AP$ is the similarity transformations of $A$. Similar matrices have the same eigenvalues and the eigenvectors are related by the following equation:

\[ V_B = P V_A, \]  \hspace{1cm} (A-18)

where $V_A$ and $V_B$ are matrices whose columns are eigenvectors of $A$ and $B$, respectively.

In order to link the eigenpairs of $A$ to those of $A_{k+1}$, we can write
\[ A_{k+1} = Q_k^T Q_{k-1}^T \cdots Q_1^T A Q_1 \cdots Q_{k-1} Q_k . \] (A-19)

which can be written in the following form:

\[ A_{k+1} = P^T A P , \] (A-20)

where \( P = Q_1 Q_2 \cdots Q_k \) is the transformation matrix which is computed in step 3. Thus the eigenvalues of \( A \) are the same as the eigenvalues of \( A_{k+1} \) (the diagonal entries of \( R_k \)). If \( A \) is symmetric, \( A_{k+1} \) converges to a diagonal matrix and the eigenvectors of a diagonal matrix are the columns of the matrix so

\[ V_A = PA_{k+1} , \] (A-21)

where \( V_A \) is a matrix whose column are the eigenvectors of \( A \).

The QR decomposition in the QR algorithm can be performed using the Modified Gram Schmidt process. The QR algorithm involves several full matrix representations and the calculation times is of \( O(n^3) \), making it impractical for matrices with size of several thousands. For large sparse matrices such as those faced in the finite element method, the Lanczos method or the subspace iteration method is used. These methods do not calculate all the eigenvalues but a desired number of the lowest eigenvalues. In addition, they don’t involve full matrices so they can be used for large matrices as well. The following section describes the subspace iteration method.
A.1.3 Subspace iteration method

The subspace iteration method can be applied to both standard and generalized eigenvalue problem. For large matrices, the method is usually used to find a relatively small number of the eigenvalue (let’s say 100 out of 100,000). Several methods are incorporated in this method; the power iteration method, the reduction of the eigenvalue problem, and the QR method. However the QR method here does not handle the original matrices but matrices of much smaller size.

Assume the generalize eigenvalue problem given below:

\[ Kx = \lambda Mx, \quad (A-22) \]

where the \( K \) and \( M \) are \( D\times D \) matrices and preferably sparse. Let \( p \) be the number of required eigenpairs (the \( p \) lowest eigenvalues) of the generalized eigenvalue problem. Choose \( m > p \) random starting vectors with length \( n \) and normalize them. \( m=2p \) is recommended. Construct \( n\times m \) matrix \( X \) whose columns are the starting vectors and iterate as follows:

Repeat the following steps for \( k=1, 2, \ldots \)

1. \( KX_{k+1} = MX_k \)
2. \( K_{k+1} = X_{k+1}^T K X_{k+1} \)
3. \( M_{k+1} = X_{k+1}^T M X_{k+1} \)
4. Find the eigenpairs of \( K_{k+1}X = \lambda M_{k+1}X \) using the QR method (\( K_{k+1} \) is a \( m\times m \) matrix) and construct matrix \( Q \) whose columns are the eigenvectors.
5. \( X_{k+1} = QX_{k+1} \)
6. If convergence achieved then the first \( p \) columns of \( X \) are the desired eigenvectors.

Step 1 corresponds to the power method but instead of acting on a single starting vector, it acts on \( m \) vectors. If only step 1 is repeated by successively substituting the current candidate in the right-hand-side and solving the matrix equation to obtain the next candidate, the columns of \( X \) converge to the eigenvectors associated with the smallest eigenvalue unless at each iteration the solutions are orthogonalized with respect to \( M \), i.e. \( X^T M X = I \). Step 2 to 5 describes the orthogonalization process. In steps 2 and 3, the original matrices \( K \) and \( M \) are projected on the space of \( X \). The projected matrices \( K_{k+1} \) and \( M_{k+1} \) are of size \( m \times m \) and can be solved for the eigenvectors efficiently using the QR method. The \( q \)-dimensional eigenvectors are projected back to the \( D \)-dimensional space in step 5. Almost all the computation time is spent on step 1, because within each iteration \( m \) large systems of equations should be solved. Although the coefficient matrix is the same for all \( m \) equations, since we use ICCG solver, the method treats them as different matrices and the computation becomes roughly proportional to \( m \). If matrix \( K \) can be inverted efficiently, the method might become considerably faster than what we have implemented in this study.

A number of practical considerations need to be made in order to reduce the unnecessary cost of calculation. For example as the iteration of the method goes on lower eigenpairs start to converge and needs to be locked such that no more improvement is required on them.
References


Publications

A. Journal Papers


B. International Conferences


