Z₃-symmetric lattice QCD simulations at finite chemical potential

By
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Abstract: The results of the Z₃-symmetric lattice QCD simulations at real finite chemical potential under phase quenched approximation are shown. As temperature increases, the expectation value of the absolute value of Polyakov-loop, which is the order parameter of confinement-deconfinement transition, abruptly increases. The transition becomes sharper when the chemical potential becomes larger.

Key word: the lattice QCD, sign problem, chemical potential, Z₃ symmetry

1. Introduction

The effects of the QCD phase transition at finite temperature (T) and finite density are important for phenomena not only in nuclear and elementary particle physics but in astrophysics and cosmology. At zero temperature, quarks are confined in hadrons and have approximately a one-third mass of the baryon. However, the lattice QCD (LQCD) simulations show that the deconfinement state of quarks with lighter current mass appears at sufficiently large T. The new state is called “quark gluon plasma (QGP)”. Now it is believed that QGP existed in the early universe after the Big Bang.

Similar deconfined state of quarks is expected when the quark number chemical potential becomes large. Such a deconfined state may exist in the interior of compact stars. However, at finite quark number chemical potential, there is a well-known sign problem and the LQCD simulations are not feasible when the chemical potential large. The sign problem is caused by the imaginary part of the effective action which appears when the quark fields are integrated out.

There is another problem in deconfinement transition. In pure gauge theory, theory has a symmetry under Z₃ transformation, and the Polyakov loop is an order parameter for the symmetry. On the other hand, the Polyakov loop is related to the free energy of an isolated heavy quark and is 0 (finite) when quark is confined (deconfined). Hence, the Z₃ symmetry is the symmetry that governs the deconfinement transition. However, the full QCD with dynamical quarks does not have the Z₃ symmetry, since the boundary condition of quark fields breaks the symmetry.

In the full QCD, there is no symmetry which distinguishes the confined phase with the deconfined one. Another important symmetry is a chiral symmetry but it is only approximate, since the current quark mass is small but finite. In fact, the LQCD results at finite temperature and zero chemical potential indicates that the transition from hadron to QGP is not a discontinuous phase transition but a continuous crossover transition [1].

However, we can modify three flavor full QCD to have the Z₃ symmetry by imposing the flavor dependent boundary condition on quark fields [2-5]. The modified theory is called Z₃-QCD. Z₃-QCD approaches the original QCD in the zero temperature limit. It is expected that the sign problem is weak in the Z₃ -symmetric theory [6]. In fact, the Z₃-symmetric 3-states and 3-dimensional Potts model has no sign problem [7]. The sign problem is weak in the Z₃-symmetric effective Polyakov loop (line) model [8].

The sign problem is expected to be weak in Z₃-QCD. At present, the lattice simulation of Z₃-QCD was done only at zero chemical potential [9]. In this brief report, we present the results of the lattice Z₃-QCD simulations at finite chemical potential under the phase quenched approximation. Note that the results under this approximation correspond to the results with isospin chemical potential rather than the ones with quark number chemical potential. The calculations of quark number density by using the reweighting method are the works in future.

This paper is organized as follows. In Sec. 2, we review the sign problem briefly and explain the phase quenched approximation. The formulation of Z₃-QCD is given in Sec. 3. In Sec. 4, numerical results are shown. Section 5 is devoted to summary and discussions.
2. Sign problem and phase quenched approximation

The grand canonical partition function in QCD at finite temperature and quark number chemical potential in path-integral form is given by

\[
Z = \int DUDq D\bar{q} \exp(-S_{QG} - S_G)
\]

\[
S_{QG} = \beta \int_0^\beta d\tau d^3 x qMq
\]

where \( q \) and \( U \) are quark and gauge (gluon) field variables, respectively, and \( \beta = 1/T \). The matrix \( M \) has the imaginary time \( \tau \), the spatial coordinate \( x \), the color, the flavor and the spinor indices, and depends on temperature \( T \) and the quark chemical potential \( \mu \). In this paper, we only show the \( \mu \)-dependence of \( M \), explicitly.

\( S_G \) is the pure gauge action the explicit form of which is not important in our discussions.

After integrating quark fields, we obtain

\[
Z = \int DU \det[M(\mu)] \exp(-S_G).
\]

The determinant satisfies the relation.

\[
\det[M(\mu)] = \det[M(-\mu^*)].
\]

This means that the determinant is not real when \( \mu \) is real and finite, and we cannot regard the integrand as a probability function. Hence, the Monte Carlo simulations with the importance sampling are not feasible.

To circumvent this problem, we use

\[
|\det[M(\mu)]| \exp(-S_G).
\]

as an approximate probability function. This approximation is called the phase quenched approximation. In two flavor case, this is equivalent to consider isospin chemical potential instead of quark number chemical potential itself. With isospin chemical potential \( \mu_i \), the determinant part is given by

\[
\det[M(\mu_i)] \det[M(-\mu_i)].
\]

The first and the second determinants represent the u quark and the d quark contributions, respectively. We obtain

\[
\det[M(\mu_i)] \det[M(-\mu_i)]
= \det[M(\mu_i)] \det[M(\mu_i)]^T
= |\det[M(\mu_i)]|^2.
\]

This is equivalent to the phase quenched approximation for the case with \( \mu \) quark number chemical potential. Using the reweighting method, in principle, the quark number density can be calculated exactly.

3. \( Z_3 \)-QCD

In pure gauge theory, there is a symmetry under the gauge transformation given by the gauge group element \( g \) with boundary condition

\[
\mu \rightarrow \mu + i\varphi \quad (\varphi = 0, 2\pi/3, -2\pi/3).
\]

Consider the QCD with exact three flavor symmetry. Suppose that the chemical potentials for each flavor (u,d,s) are given by

\[
\mu_u = \mu + i2\pi/3,
\mu_d = \mu - i2\pi/3,
\mu_s = \mu.
\]

It is clear that this system is invariant under the gauge transformation above. This is a \( Z_3 \)-symmetric QCD, namely, \( Z_3 \)-QCD \([2-9]\). For technical reasons, in actual calculations, we consider six flavor case and consider the phase quenched approximation. Furthermore, we multiply the quark action by the factor 1/3. Obtained theory approaches to the original two-flavor QCD in zero temperature limit, since boundary conditions are not relevant in the limit.

4. Numerical results

In this study, we used the same lattice action and the same parameter settings as the ones used in Ref. \([10]\). We used a lattice with the spatial size \( 8^3 \) and the temporal size 4.

We used the renormalization-group-improved Iwasaki gauge action \([11]\) and the two flavor Wilson fermion action. We done the simulations by using the Hybrid Monte Carlo program based on the Lattice QCD Tool Kit \([12]\). (For the detail information, See Ref. \([10]\).) For comparison, we also done the simulations in the case of ordinary two flavor QCD.

Figure 1 shows the \( \beta \)-dependence of the absolute value of the Polyakov-loop \( P \) in the case of ordinary two flavor QCD at \( \mu = 0 \), where \( \beta = 6/g^2 \) with the coupling
constantly increasing smoothly as \( \beta \) increases. This behavior indicates that the deconfinement transition is crossover as already known [1].

Fig. 1 The \( \beta \)-dependence of the absolute value of the Polyakov-loop in the case of the ordinary two flavor QCD at \( \mu=0 \).

Figure 2 shows the same as Fig. 1 but for \( Z_3 \)-QCD. The absolute value of the Polyakov-loop increases rapidly above \( \beta=2.1 \). This behavior indicates that there is a first order phase transition as was already seen in Ref. [9].

Fig. 2 \( \beta \)-dependence of the absolute value of the Polyakov-loop in the case of \( Z_3 \)-QCD at \( \mu=0 \).

Figure 3 shows the same as Fig. 1 but the one at \( \mu/T=1.0 \). Note that this situation corresponds to the case with finite isospin chemical potential rather than the one with quark chemical potential, since we used the phase quenched approximation. As in the case with zero chemical potential, the absolute value of the Polyakov-loop increases smoothly.

Fig. 3 \( \beta \)-dependence of the absolute value of the Polyakov-loop in the case of the ordinary two flavor QCD at \( \mu/T=1.0 \).

Figure 4 shows the same as Fig. 2 but the one at \( \mu/T=1.0 \). As in the case with zero chemical potential, the absolute value of the Polyakov-loop increases rapidly above \( \beta=2.1 \). It seems that the transition becomes sharper than the one at zero chemical potential.

Fig. 4 \( \beta \)-dependence of the absolute value of the Polyakov-loop in the case of \( Z_3 \)-QCD at \( \mu/T=1.0 \).

Figure 5 shows the same as Fig. 4 but for number density. Note that this number density corresponds to the isospin number density rather than quark number density, since we used the phase quenched approximation. As in the case of the Polyakov-loop, the number density increases rapidly above \( \beta=2.1 \).

Fig. 5 \( \beta \)-dependence of the absolute value of the Polyakov-loop in the case of \( Z_3 \)-QCD at \( \mu/T=1.0 \).

5. Summary and discussions

In this brief report, we present the results of the lattice \( Z_3 \)-QCD simulation at finite chemical potential, using the phase quenched approximation.

In \( Z_3 \)-QCD, the absolute value the Polyakov-loop has a rapid jump as \( \beta \) increases, while the quantity increases smoothly in the ordinary LQCD. In \( Z_3 \)-QCD, the transition becomes sharper when the chemical potential becomes large.
Fig. 5 β-dependence of the absolute value of the number density in the case of $Z_3$-QCD at $\mu/T=1.0$. a is a size of lattice spacing.

The results in this report correspond to the ones at finite isospin chemical potential rather than finite quark number density, since the phase quenched approximation was used. However, we can obtain the results at finite quark number density by using the reweighting method. It is a future work.

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References