A Vertical Diffusion Model of Replacement of Salt Water by Fresh Water

Hiroyuki Cho
(Laboratory of Irrigation and Drainage Engineering)
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Summary

A model experiment was performed by using an annular tank to investigate the rate of decrease of salt concentration in water by replacing salt water with an inflow of fresh water and drainage of the mixed water at a reservoir separated from the sea. The experiment was done with a very small scale of turbulent flow in the water layer, so that the rate of salt decrease could be explained by a diffusion equation with a constant diffusion coefficient as an overall mean value in the water. As a vertical diffusion model, an exact solution of the equation was obtained satisfying the initial and boundary conditions approximating the experimental conditions, and it was shown that salt concentration distribution temporally converges into a figure with a certain gradient of salt concentration caused by the situations in which the amount of salt flowing out is replaced by salt from the bottom sediment. In comparison of calculations of salt concentration distribution with actual measurements at the end of experiment, the salt replacement proceeded much faster than the calculations predicted.

Key words: diffusion equation, salt concentration, exact solution, vertical diffusion model

1. Introduction

At a reservoir separated from the sea by an enclosure dam in a sea reclamation area, the salt water encased was gradually replaced through the action of mixing and drainage by fresh water from rivers and also supplied by precipitation. In order to explain the process of the changing of salt concentration during the desalinization, we simplified the process and assumed that as fresh water intruded over the surface of salt water initially reserved, the salt in the water would be diffused in or mixed with the fresh water due to the turbulence occurring from water flow, and the mixed water would be drained out of the reservoir. We also assume that the salt contained in the bottom sediment would be released continuously to the upper water area.

We considered that salt is not transported due to diffusion when the rate of fresh water intrusion is much smaller than the velocity of the salt water. From calculations of the changing salt concentration due to turbulent diffusion, we deduced the exact solution of the diffusion equation with the boundary conditions which show that the salt concentration of surface water changes temporally and salt is supplied from the bottom sediment continuously. Experiments were performed using an annular tank as a model of a reservoir.
2. Experiment with an annular tank

The experiment was performed by using an annular tank which is shown in detail in Fig. 1. To provide salt concentration release from the bottom sediment, glass beads (diameter; 0.15 to 0.25 mm) saturated with salt water were used to cover the bottom of the tank. The thickness of the glass beads layer was adjusted to 15 cm and its mean porosity equaled about 0.40. Water depth was 10.6 cm to the surface of the bottom layer. A micro-tube pump was fixed at the water surface level to supply fresh water through an inlet tube. A water outlet hole was drilled on the other side of the fresh water inlet to drain mixing water. Salt concentration was measured by an E.C.Meter (Electric Conduction Meter) and recorded by a Data Logger. E.C.Sensors with 4 mm diameter were set at the distances, 0.5 cm, 2.7 cm, 4.3 cm, 5.7 cm, 7.2 cm and 8.8 cm from the surface of the bottom layer. A rotating screen was equipped to generate water flow, and its lowest part was positioned at 0.5 cm below the water surface. Velocity distribution was observed by means of photographic tracing of stream lines.

3. Experiment procedure and results

The bottom glass beads layer was saturated with salt water concentrated to 2% salinity. Salt water was added up to the level of the water outlet, then fresh water was pumped in the rate of 0.504 cm³/s. Simultaneously, the screen began to rotate at a speed of 0.76 r.p.m. (3.3 cm/s). Water was drained from the water outlet at the same rate as fresh water entered. E.C.Meter measurements were carried out at 30 minute intervals and started at the same time as the beginning of screen rotation. The experiment was continued for about four days.

The temporal change of salt concentration measured by each sensor is shown in Fig. 2. Variations of salt cononcentration with time at various height of z.

![Fig. 1 Equipment layout for experiments on diffusion phenomena.](image)

![Fig. 2 Variations of salt concentration with time at various height of z.](image)
2 and \( z \) expresses height from the surface of glass beads layer. \( C_0 \) means initial salt concentration of salt water in the annular tank and \( H \) represents water depth. Fig. 3 shows that values measured at the position of \( z = 8.8 \) cm agree well with those of the water drained from outlet, and both of them decrease exponentially with time. Fig. 4 reveals velocity distribution in the water and it seems to be close to the Couette flow.

4. Vertical salt concentration diffusion model

A model explaining the process of replacement of salt water by fresh water is assumed as shown schematically in Fig. 5. It means that a closed water area initially stores salt water (salt concentration of \( C_0 \)), salt concentration near the water surface changes temporally by inflow of fresh water and outflow of mixed water, and mixing is performed by turbulent diffusion based on the vertical difference of salt concentration and turbulence. Through experiment, we found that the salt concentration of surface water decreases exponentially with time as shown in Fig. 6. The thickness of bottom sediment layers in

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![Image 1](image1)

Fig. 3  Comparison of salt concentration at \( z = 8.8 \) cm (below 1.8 cm from water surface) with that of water surface.

![Image 2](image2)

Fig. 4  Normalized velocity profiles.

![Image 3](image3)

Fig. 5  Diffusion model and boundary conditions.

![Image 4](image4)

Fig. 6  Approximated values of salt concentration at water surface by the form of \( \exp(-Gt) \).
actual reservoirs separated from the sea by an enclosure dam reach about 10 to 20 meters. Pores in the sediment are saturated with salt water and salt diffuses from inside of the layer to the outside at a rate of the order of $10^{-5}$ cm$^2$/s, and therefore it enables us to consider that salt was supplied continuously from the bottom layer to the upper water layer during the experiment described in this paper, and so salt concentration at $z=0$ would remain $C_0$, the same as the initial salt concentration.

The kinematic diffusion equation satisfying conditions mentioned above was established to obtain an exact solution considered as a model of the replacement process expressed by the experiment, but the vertical kinematic diffusion coefficient in this equation is assumed to be constant as representing the overall mean value in the water layer.

The equation is as follows. The $z$ axis is taken vertically in the direction of the water surface.

$$\frac{\partial C}{\partial t} = K_z \frac{\partial^2 C}{\partial z^2}$$

The initial condition and boundary conditions are as follows

$$t = 0, \ 0 \leq z \leq H; \ C = C_0$$

$$t > 0, \ z = 0; \ C = C_0$$

$$t > 0, \ z = H; \ C = C_0 \exp(-Gt)$$

We apply the Laplace transformation to (2)

$$K_z \frac{d^2 U}{dz^2} - sU = -C_0$$

then

$$L\{C(z, t)\} = U(z, s)$$

The solution of (5) is

$$U = \frac{C_0}{s} + A \cdot \exp(z\sqrt{s}/K_z) + B \cdot \exp(-z\sqrt{s}/K_z)$$

We applied the Laplace transformation to (3), (4)

$$U(0, s) = \frac{C_0}{s}$$

$$U(H, s) = \frac{C_0}{G+s}$$

$$U(0, s) = A + B + \frac{C_0}{s} = \frac{C_0}{s}$$

$$\therefore \ A = -B$$

$$U(H, s) = \frac{C_0}{s} + A\{\exp(H\sqrt{s}/K_z) - \exp(-H\sqrt{s}/K_z)\} = \frac{C_0}{G+s}$$

$$\therefore \ A = \frac{C_0}{G+s} - \frac{C_0}{s} \} / \{\exp(H\sqrt{s}/K_z) - \exp(-H\sqrt{s}/K_z)\}$$

From (6), (9) and (10), $U(z, s)$ is given by the following equation.

$$U(z, s) = \frac{C_0}{s} \left\{ \frac{C_0}{G+s} - \frac{C_0}{s} \right\} \frac{\exp(z\sqrt{s}/K_z) - \exp(-z\sqrt{s}/K_z)}{\exp(H\sqrt{s}/K_z) - \exp(-H\sqrt{s}/K_z)}$$

$$= \frac{C_0}{s} \left\{ \frac{G\sqrt{s}}{s(G+s)} \right\} \frac{\sinh(z\sqrt{s}/K_z)}{\sinh(H\sqrt{s}/K_z)}$$

(1)
We determined $C(z, t)$ by the use of the Inversion Theorem for the Laplace tranformation and it was shown that

$$
C(z, t) = C_0 - \frac{GC_0}{2\pi i} \lim_{\gamma \to \infty} \int_{\gamma - i\infty}^{\gamma + i\infty} \exp(at) \frac{\sinh(z\sqrt{a/K_x})}{a(G + a)} \sinh(H\sqrt{a/K_x}) \, da
$$

(12)

Here,

$$
\frac{GC_0}{a(G + a)} \frac{\sinh(z\sqrt{a/K_x})}{\sinh(H\sqrt{a/K_x})}

= \frac{GC_0}{a(G + a)} \frac{z\sqrt{a/K_x} + (z\sqrt{a/K_x})^3/3! + \cdots}{H\sqrt{a/K_x} + (H\sqrt{a/K_x})^3/3! + \cdots}

= \frac{GC_0}{a(G + a)} \frac{z/H + ((z^3/H)(a/K_x)^3/3!) + \cdots}{1 + (H/\sqrt{a/K_x})^3/3! + \cdots}

= f(a, z)
$$

(13)

Isolated singular points are

$$
a = 0, a = -G, a = -\frac{Kn^2\pi^2}{H^2} \quad (n = 1, 2, \cdots)
$$

(14)

From (13), it is evident that

$$
\lim_{a \to 0} [\exp(at) \cdot a \cdot f(a, z)] = \frac{C_0}{H} z
$$

(15)

Thus, $f(a, z)$ has a first-order pole with the residue of $(C_0/H)z$ at $a = 0$ and

$$
\lim_{a \to 0} [\exp(at) \cdot (a + G) \cdot f(a, z)]

= -C_0 \exp(-Gt) \frac{\sinh(z\sqrt{G/K_x})}{\sinh(iH\sqrt{G/K_x})}

= -C_0 \exp(-Gt) \frac{\sin(z\sqrt{G/K_x})}{\sin(H\sqrt{G/K_x})}
$$

(16)

$f(a, z)$ has a first-order pole at $a = -G$, and the residue is

$$
-C_0 \exp(-Gt) \frac{\sin(z\sqrt{G/K_x})}{\sin(H\sqrt{G/K_x})}
$$

Now, we put the branch point for $a$, along the positive real axis so that $a$ is analytic along the negative real axis. Then, $f(a, z)$ has a fractional form of $p(a, z)/q(a, z)$, and so

$$
p(a, z) = \frac{GC_0}{a(G + a)} \sinh(z\sqrt{a/K_x})
$$

(17)

$$
q(a, z) = \sinh(H\sqrt{a/K_x})
$$

(18)

$p(a, z)$ and $q(a, z)$ are analytic at

$$
a = -\frac{Kn^2\pi^2}{H^2}
$$

then,

$$
q(-\frac{Kn^2\pi^2}{H^2}, z) = \frac{H^2}{2\sqrt{K_x}} \sqrt{-\frac{1}{(Kn^2\pi^2)/(H^2)}} \cosh[H\sqrt{-(Kn^2\pi^2)/(K_xH^2)}]

= \frac{H^2}{2K_x} \frac{\cos(n\pi)}{n\pi} \neq 0
$$

(19)

Thus,

$$
a = -\frac{Kn^2\pi^2}{H^2}
$$

are first-order poles, and the residue for $\exp(at) \cdot f(a, z)$ is given by
\[
\frac{p(-K_z n^2 \pi^2)/(H^2), z)}{q(-K_z n^2 \pi^2)/(H^2)} \exp(-\frac{K_z n^2 \pi^2 t}{H^2}) = \frac{2K_z}{H^2} n \pi \frac{n \pi i}{\cos(n \pi)} \\
\cdot \left(\frac{G}{(G - (K_z n^2 \pi^2)/(H^2))} \right) \sinh(z \sqrt{(G/(K_z n^2 \pi^2)/(H^2))} \exp(-t(K_z n^2 \pi^2)/(H^2)) \\
= \frac{(-1)^n 2G \sin((z/H)n \pi)}{n \pi \left(G - (K_z n^2 \pi^2)/(H^2)\right)} \exp(-t(K_z n^2 \pi^2)/(H^2))
\]

(20)

The integral in (2) is given by the sum of residues for poles of (4). Therefore, from (16), (17) and (20), we have finally

\[
C(z, t) = C_0 \left(1 - \frac{z}{H} + \frac{\sin(z \sqrt{G/K_z})}{\sin(H \sqrt{G/K_z})} \exp(-Gt) + \frac{2G}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sin((z/H)n \pi)}{n(G - (K_z n^2 \pi^2))/(H^2)} \right) \exp(-t(K_z n^2 \pi^2)/(H^2))
\]

(21)

as a exact solution of (1).

A value of \( G \) in (21) should be decided by the actual salt concentration distribution. When \( G \) is defined, salt concentration distribution changes according with \( K_z \) values.

5. Results and discussion

As a result of the experiment, the temporal change of salt concentration at the water surface could be approximated by \( \exp(-Gt) \) and we gave \( 2.0 \times 10^{-5} \) as \( G \) value and it is shown in Fig. 6. Eq. (21) is calculated on the points of \( z=0.5, 2.7, 4.3, 5.7, 7.7, 8.8 \) (cm) in the case of \( K_z=1.0 \times 10^{-4} \text{ cm}^2/\text{s} \) and those results are expressed in Fig. 7. On calculat-

![Fig. 7 Calculated salt concentration by Eq. (21) in the case of \( K_z=1.0 \times 10^{-4} \text{ cm}^2/\text{s} \).](image)

![Fig. 8 Calculated salt concentration by Eq. (21) in the case of \( K_z=5 \times 10^{-5}, 1 \times 10^{-4}, 1 \times 10^{-5} \text{ cm}^2/\text{s} \) at \( z/H=0.68 \).](image)

![Fig. 9 Comparison of salt concentration distribution between those calculated and those observed at the end of experiment.](image)
ing (21), 2 is used as \( n \) because (21) is able to converge to the condition not exceeding permissible error which is 1/1000 for \( C \) values when \( n \) is more than 2. Fig. 8 shows values calculating (21) in the case of \( K_s = 5 \times 10^{-5}, 1 \times 10^{-4}, 1 \times 10^{-3} \) (cm\(^2\)/s) for \( z/H = 0.68 \). Those figures obviously reveal the temporal change of \( C \) distribution for each \( K_s \) or \( z/H \). All solid lines converge to certain \( C \) values which are bigger than that of the surface water and it means that a salt concentration gradient exists on the condition that the quantity of salt supplied from bottom layer is balanced by that drained from outlet. It may be deduced from the Eq. (21) that salt concentration distribution approaches following equation asymptotically with time

\[
C(z) = C_0(1 - z/H)
\]  

(22)

On the other hand, salt concentration distribution measured by the experiment is shown in Fig. 2 and it designates the appearance of the mixed layer in which the salt concentration gradient disappeared and the value of salt concentration is the same as that at water surface. The thickness of the mixed layer tends to develop downwards from the water surface with time. Fig. 9 shows the comparison of salt concentration distribution at the end of experiment between that calculated and what was actually observed. It appears that;

(1) Boundary conditions which were assumed to solve the diffusion equation were nearly satisfactory during the experiment.

(2) The total amount of salt expressed by oblique lines in Fig. 9 was not actually supplied from the bottom layer to water layer.

(3) The phenomena appering in this experiment may be defined as a two layered density flow (though the velocity distribution was continuous through two layers).

We observed that continual salt transportation from the lower level to the upper level of the water, which is designated by the exact solution (21) with an overall diffusion coefficient, did not occur in this experiment as we had calculated, and so desalinization of the water proceeded without the salt supply from the bottom sediment and the actual mechanism of desalinization was quite different from what early calculations had indicated.

The experiment in this paper was performed under almost the same hydraulic conditions as the one of Cho et al\(^1\), which was carried out to clarify the mechanism of salt diffusion from the bottom sediment layer to the upper water layer, but that didn't deal with fresh water inflow. It concluded that the temporal change of salinity distribution could be revealed by using a vertical diffusion equation with a constant diffusion coefficient explained as an overall diffusion coefficient in the water. As the results of the experiment in this paper show, replacement of salt water by fresh water wasn't proceeded by overall homogeneous diffusion but by an increase in the mixing layer's thickness which was explained as lowering density interface by Tamai\(^2\) and so on\(^3, 4\). If the velocity gradient near the water surface were less and the turbulent mixing more gentle at the initial stage of the experiment, the replacement process could proceed as a diffusion phenomena expressed by (21). It seems to depend upon critical hydraulic conditions which remain uncertain, and from now on we should clarify what conditions in the process causes two
-layered density flow.

References

淡水化過程における鉛直拡散モデル

長 裕幸
（生産情報科学講座）
平成2年5月31日受理

摘 要

干拓貯水池等において初期に湛水していた塩水が淡水の流入及び塩水の域外への排除によって淡水化していく過程について、回流水槽を用いたモデル実験を行った。実験では非常に拡散規模の小さい現象に対して、底質からの塩分の補給が存在する場合の塩分濃度の鉛直分布について、その時間的な変化を調べた。また、実験で得られた境界条件、初期条件を満足し、水域においてオーバーオールな定拡散係数が存在するという仮定のもとで、鉛直1次元の拡散方程式の解析解を求め検討を行った。その結果、定拡散係数の場合、時間の経過に伴い域外への塩分の流出と底質からの塩分の補給がバランスする一定の濃度勾配に漸近していく形で淡水化が進行することがわかった。また、実験終了時における塩分濃度分布を比較すると、計算値に比べ実験ではかなり淡水化が進んでいた。