Effective potential in quantum hadrodynamics

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Abstract: The density-dependent effective potential in nuclear matter is studied by using the quantum hadrodynamics and the nonperturbative evolution equation. The result beyond the ordinary mean-field approximation, which is consistent with the thermodynamical relations, is obtained.

1 Introduction

In the past quarter century, the properties of nuclei and nuclear matter have been widely studied by the quantum hadrodynamics (QHD). Usually the mean field (or Hartree) approximation is used to calculate the medium effects. Naturally, there are many attempts to include the higher-order corrections in the calculations, e.g., the Hartree-Fock approximation, the relativistic Dirac-Brueckner-Hartree-Fock calculations and the nuclear Schwinger-Dyson calculations. However, it is very difficult to retain the thermodynamical consistency in these calculations.

Recently, Kouno et al. applied the nonperturbative renormalization group equations to QHD. It seems that, in this approach, the thermodynamical consistency may be able to be checked more easily, since the effective potential itself is directly treated in this formalism.

There are many types of the nonperturbative renormalization group equations or the non-perturbative evolution equations for the effective action. Berges et al. proposed the chemical potential flow equation for the density-dependent effective action at finite density. This equation connects the effective action at zero-density and the one at the finite density. The consistency with the thermodynamical relations is trivial in this formalism.

In this brief report, we study the density-dependent effective potential (energy density) in nuclear matter, by using QHD and the Legendre transformed version of the nonperturbative evolution equations proposed by Berges et al. This paper is organized as follows. In Sec.2, we review the nonperturbative evolutions equation briefly. In Sec.3, we study the density-dependent effective potential (energy density) in nuclear matter using the evolution equation. The result which is beyond the ordinary mean-field approximation and is consistent with the thermodynamical relations is obtained. The Sec.4 is devoted to the summary.
2 Evolution equations

In this section, we review the nonperturbative evolution equations. In the exact renormalization group method, the nonperturbative evolution equation for the effective action is obtained as follows. Consider the effective action which contain only the high energy quantum fluctuation effects

\[ \exp (-W_{A}[J]) = \int D\phi \exp (-S[\phi] + \int J \cdot \phi - S_{A}[\phi]), \]  \hspace{1cm} (1)

where \( \phi \) is the quantum field and

\[ \int J \cdot \phi = \int d^{4}x J(x) \phi(x). \]  \hspace{1cm} (2)

The S. \( \phi \) is given by

\[ S_{A}[\phi] = \frac{1}{2} \int \frac{d^{4}q}{(2\pi)^{4}} \phi(-q) R_{\Lambda}(q) \phi(q), \]  \hspace{1cm} (3)

where \( R_{\Lambda} \) is the cutoff function which suppress the quantum effects below the cutoff \( \Lambda \). The expectation value of \( \phi \) in the presence of the cut off \( S_{A} \) and the source \( J \) is given by

\[ \phi(x) \equiv \langle \phi(x) \rangle = \frac{\delta W_{A}[J]}{\delta J(x)}. \]  \hspace{1cm} (4)

Define the effective action in the presence of \( S_{A} \) and \( J \) as

\[ \Gamma_{A}[\phi] = -W_{A}[J] + \int d^{4}x J(x) \phi(x) - S_{A}[\phi], \]  \hspace{1cm} (5)

and

\[ \hat{\Gamma}_{A}[\phi] \equiv \Gamma_{A}[\phi] + S_{A}[\phi] = -W_{A}[J] + \int d^{4}x J(x) \phi(x). \]  \hspace{1cm} (6)

One obtains

\[ \frac{\partial}{\partial t} \hat{\Gamma}_{A}[\phi_{\text{fixed}}] = -\left( \frac{\partial W_{A}}{\partial t} \right)[J_{\text{fixed}}] - \int d^{4}x \frac{\delta W_{A}}{\delta J(x)} \frac{\partial J(x)}{\partial t} + \int d^{4}x \phi(x) \frac{\partial J(x)}{\partial t}, \]  \hspace{1cm} (7)

where \( A = \Lambda_{A} e^{-i t} \). The first term in the right hand side (r. h. s.) in Eq. (7) is given by

\[ \left( \frac{\partial W_{A}}{\partial t} \right)[J] = \langle \frac{\partial}{\partial t} S_{A}[\phi] \rangle = \langle \frac{1}{2} \int d^{4}x d^{4}y \phi(x) R_{\Lambda}(x,y) \phi(y) \rangle \]

\[ = \frac{1}{2} \int d^{4}x d^{4}y \phi(x) \phi(y) \frac{\partial}{\partial t} R_{\Lambda}(x,y), \]  \hspace{1cm} (8)

where \( R_{\Lambda}(x,y) \equiv (-\partial_{x}^{2}) \delta(x-y) \). If one define the connected two-point green function as

\[ G(x,y) = \langle \phi(x)\phi(y) \rangle - \langle \phi(x) \rangle \langle \phi(y) \rangle, \]  \hspace{1cm} (9)

one obtains

\[ \frac{\partial}{\partial t} \hat{\Gamma}_{A}[\phi] = \frac{1}{2} \int d^{4}x d^{4}y \{ G(x,y) \frac{\partial}{\partial t} R_{\Lambda}(x,y) + \phi(x) \frac{\partial}{\partial t} R_{\Lambda}(x,y) \phi(y) \} \]

\[ = \frac{1}{2} \text{Tr} \left( G \frac{\partial}{\partial t} R_{A} \right) + \frac{\partial}{\partial t} S_{A}[\phi]. \]  \hspace{1cm} (10)

Finally, one obtains

\[ \frac{\partial}{\partial t} \Gamma_{A}[\phi] = \frac{1}{2} \text{Tr} \left( G \frac{\partial}{\partial t} R_{A} \right) = \frac{1}{2} \text{Tr} \left( \Gamma_{A}^{(2)}[\phi] + R_{A} \right)^{-1} \frac{\partial}{\partial t} R_{A}, \]  \hspace{1cm} (11)
where
\[
(\Gamma^{(2)}_A(q,q^*) = \frac{\delta^2 \Gamma_A}{\delta \phi(-q) \delta \phi(q^*)}.
\] (12)

Similarly, various evolution equations are obtained by differentiations with respect to the continuous parameter contained in the partition function. Next consider the partition function at the finite chemical potential \( \mu \). Define
\[
\exp \left( W[J] \right) = \int D\phi \exp \left( -S[\phi] - S_\mu[\phi] + \int J \cdot \phi \right)
\] (13)

and
\[
\Gamma[\phi] = -W[J] + \int J \cdot \phi - S_\mu[\phi], \quad \varphi = \frac{\delta W}{\delta J},
\] (14)
where the field \( \varphi \) includes the fermion field \( \phi \) and
\[
S_\mu[\phi] = \int d^4x \bar{\psi}(x) \gamma_0 \psi(x).
\] (15)

Differentiating Eqs. (13) and (14) with respect to the chemical potential \( \mu \), one obtains
\[
-\frac{\partial}{\partial \mu} \Gamma[\varphi] = \int d^4x \bar{\psi}(x) \gamma_0 \psi(x),
\] (16)
where the r. h. s. in Eq. (16) is the baryon number of the system and can be written in term of \( \Gamma^{(2)}[\varphi] \). Similarly, differentiating \( W[J] \) and \( \Gamma[\varphi] \) with respect to the square of the bare boson mass, one obtains
\[
\frac{\partial}{\partial \mu} \Gamma[\varphi] = -\frac{1}{2} \int d^4x \bar{\psi}(x) \gamma_0 \psi(x),
\] (17)
where \( \varphi \) is the boson field.

Differentiation with respect to fermion mass \( M \) yields
\[
\frac{\partial}{\partial M} \Gamma[\varphi] = \int d^4x \bar{\psi} \gamma_0 \psi.
\] (18)

The r. h. s. in Eq. (18) is the scalar number of fermions. If \( S \) include the gauge fixing term
\[
\frac{1}{2} \xi (\partial_\mu \chi_\mu)^2,
\]
differentiation with respect to \( \xi \) yields
\[
\frac{\partial}{\partial \xi} \Gamma[\varphi] = -\frac{1}{2} \int d^4x (\partial_\mu \chi_\mu(x))^2 = -\frac{1}{2} \int d^4x \int \frac{d^4k}{(2\pi)^4} k_\mu k_\nu G_{\mu\nu}(k),
\] (19)
where \( G_{\mu\nu} \) is the full propagator for the field \( x \). The r. h. s. in Eq. (19) becomes trivially zero if \( k_\mu G_{\mu\nu}(k) = 0 \). Therefore, the Landau gauge is the fixed point for the evolution equation.

3 Effective couplings and equation of state of nuclear matter

In this section, using the QHD and the evolution equation proposed by Berges et al., we study the density-dependent effective potential(energy density)in nuclear matter. Consider the following effective Lagrangian at finite density :
\[
L = \bar{\psi} \left\{ i \partial_\mu + \sum_\sigma \left( \rho_\sigma, \omega_\sigma \right) \right\} \phi - \left\{ M + \left( \rho_\sigma, \omega_\sigma \right) \right\} \phi
\]

\[
+ \frac{1}{2} \partial_\mu \phi \partial_\mu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - U_M(\rho, \sigma, \omega),
\] (20)

where \( \phi, \sigma, \omega_\mu \) are the nucleon field, the \( \sigma \)-meson field and \( \omega \)-meson field, respectively, and \( F_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu \). The \( \rho \) is the baryon density. The \( U_M \) is the potential which includes only the meson field. Since we use the evolution equation (16) with the local potential approximation, we
have neglected the derivative couplings interactions in the effective Lagrangian\(^{(20)}\).

In the uniform nuclear matter, the ground-state expectation value of the spatial component of the \(\omega\)-meson field is zero. Therefore, below, we only work with the expectation value \(\langle \omega^0 \rangle\) of the time component of the \(\omega\)-meson field and write it in the symbol of \(\omega\). We also write the expectation value \(\langle \sigma \rangle\) in the symbol of \(\sigma\).

The \(\Sigma_\Sigma\) and \(\gamma^\nu \Sigma_\nu\) are the self-energies of the nucleon. Since \(\Sigma_i^\prime\) (\(i = 1, 2, 3\)) has at least one spatial component \(\omega\)-field, it also becomes zero in our approximation. Below, we write \(\Sigma^0\) as \(\Sigma_\nu\). In the effective Lagrangian\(^{(20)}\), we have neglected the other parts of the self-energies but all of them also vanish in our approximation. (For example, the tensor part \(\bar{\psi} [\gamma^\mu, \gamma^\nu] \Sigma_{\mu\nu} \psi\) vanishes, since \(\Sigma_{\mu\nu}\) is antisymmetric in the subscripts \(\mu\) and \(\nu\) and includes at least one \(\omega^i\).)

For uniform nuclear matter, Eq. \(^{(16)}\) can be written as

\[
\frac{\partial P(\mu, \sigma, \omega)}{\partial \mu} = \rho(\mu, \sigma, \omega), \tag{21}
\]

where \(\sigma\), \(\omega\) and \(\rho\) are the expectation value of \(\sigma\) meson field, the expectation value of the time-component of the \(\omega\) meson field and the baryon density, respectively.

The equation \(^{(21)}\) is somewhat different from the thermodynamical relation

\[
\frac{dP(\mu, \sigma, \omega)}{d\mu} = \rho(\mu, \sigma, \omega), \tag{22}
\]

However, these two equations are equivalent, namely,

\[
\frac{dP(\mu, \sigma, \omega)}{d\mu} = \frac{\partial P(\mu, \sigma, \omega)}{\partial \mu} + \frac{\partial P(\mu, \sigma, \omega)}{\partial \sigma} \frac{d\sigma}{d\mu} + \frac{\partial P(\mu, \sigma, \omega)}{\partial \omega} \frac{d\omega}{d\mu} \tag{23}
\]

because of the equations of motion for the \(\sigma\) and \(\omega\)

\[
\frac{\partial P(\mu, \sigma, \omega)}{\partial \sigma} = 0 \quad \text{and} \quad \frac{\partial P(\mu, \sigma, \omega)}{\partial \omega} = 0. \tag{24}
\]

It is difficult to use Eq. \(^{(21)}\) directly, since \(\rho\) is not a unique function of \(\mu\) in nuclear matter. Therefore, we perform the Legendre transformation

\[
\varepsilon(\rho, \sigma, \omega) = \mu \rho - P(\mu, \sigma, \omega). \tag{25}
\]

We obtain the equation of motion for \(\sigma\)-meson as

\[
\frac{\partial \varepsilon(\rho, \sigma, \omega)}{\partial \sigma} = \frac{\partial \mu(\rho, \sigma, \omega)}{\partial \rho} \rho - \frac{\partial \mu(\rho, \sigma, \omega)}{\partial \sigma} - \frac{\partial P(\mu, \sigma, \omega)}{\partial \sigma} = 0. \tag{26}
\]

Similarly, we obtain the equation of motion for \(\omega\)-meson as

\[
\frac{\partial \varepsilon(\rho, \sigma, \omega)}{\partial \omega} = 0. \tag{27}
\]

We obtain

\[
\frac{\partial \varepsilon(\rho, \sigma, \omega)}{\partial \rho} = \frac{\partial \mu(\rho, \sigma, \omega)}{\partial \rho} \rho + \frac{\partial \mu(\rho, \sigma, \omega)}{\partial \rho} \frac{\partial P(\mu, \sigma, \omega)}{\partial \mu} = \mu(\rho, \sigma, \omega). \tag{28}
\]
Again, this evolution equation is equivalent to the thermodynamical relation
\[
\frac{d\varepsilon (\rho, \sigma, \omega)}{d\rho} = \mu (\rho, \sigma, \omega),
\]
(28)
because of the equations of motion \(26\) and \(27\).

Below, we solve the evolution equation \(28\) using a simple approximation. For the nucleon self-energy \(\Sigma\), we assume
\[
\Sigma_s (\rho, \sigma, \omega) = \Sigma_{s1} (\sigma, \omega) + k_F \Sigma_{s2} (\sigma, \omega)
\]
and
\[
\Sigma_v (\rho, \sigma, \omega) = \Sigma_{v1} (\sigma, \omega) + k_F \Sigma_{v2} (\sigma, \omega).
\]
(30)
(31)
In this approximation, the chemical potential \(\mu\) reads
\[
\mu (\rho, \sigma, \omega) = \sqrt{1 + \Sigma_{s2}^2} \sqrt{k_F^2 + \dot{M}^2} - \Sigma_s,
\]
where
\[
\dot{M} = \frac{M + \Sigma_{s1}}{1 + \Sigma_{s2}}
\]
and
\[
\dot{k}_F = k_F + \frac{(M + \Sigma_{s1}) \Sigma_{s2}}{1 + \Sigma_{s2}} = k_F + \dot{M} \Sigma_{s2}.
\]
(33)
(34)
Differentiating the nucleon self-energy with respect to the meson-field expectation value, we obtain the effective meson-nucleon couplings, namely,
\[
g^s_{\sigma \omega} = - \frac{\partial \Sigma_s (\rho, \sigma, \omega)}{\partial \sigma} = - \frac{\partial \Sigma_{s1} (\sigma, \omega)}{\partial \sigma} - \frac{\partial \Sigma_{s2} (\sigma, \omega)}{\partial \sigma} k_F,
\]
\[
g^s_{\omega \sigma} = - \frac{\partial \Sigma_s (\rho, \sigma, \omega)}{\partial \omega} = - \frac{\partial \Sigma_{s1} (\sigma, \omega)}{\partial \omega} - \frac{\partial \Sigma_{s2} (\sigma, \omega)}{\partial \omega} k_F,
\]
\[
g^v_{\omega \sigma} = - \frac{\partial \Sigma_v (\rho, \sigma, \omega)}{\partial \sigma} = - \frac{\partial \Sigma_{v1} (\sigma, \omega)}{\partial \sigma} - \frac{\partial \Sigma_{v2} (\sigma, \omega)}{\partial \sigma} k_F,
\]
and
\[
g^v_{\sigma \omega} = - \frac{\partial \Sigma_v (\rho, \sigma, \omega)}{\partial \omega} = - \frac{\partial \Sigma_{v1} (\sigma, \omega)}{\partial \omega} - \frac{\partial \Sigma_{v2} (\sigma, \omega)}{\partial \omega} k_F.
\]
(35)
(36)
(37)
(38)
We can also integrate the evolution equation \(28\) analytically and obtain
\[
\varepsilon (\rho, \sigma, \omega) = \sqrt{1 + \Sigma_{s2}^2} \left( \varepsilon_N (\dot{k}_F, \dot{M}) - \varepsilon_N (\dot{M} \Sigma_{s2}, \dot{M}) \right)
\]
\[
- \rho \Sigma_v - \frac{3}{4} \left( \frac{3\pi^2}{\lambda} \right)^{\frac{1}{3}} \rho^{\frac{4}{3}} \Sigma_{v2} + U_M (\rho = 0, \sigma, \omega),
\]
(39)
where \(\lambda = 2\) in symmetric nuclear matter and
\[
\varepsilon_N (x, y) = \frac{\lambda}{12\pi^2} \left[ x \sqrt{x^2 + y^2} - \frac{3}{2} y^2 \right] - \frac{3}{2} y^4 \log \left( \frac{x^2 + y^2 + 1}{y} \right).
\]
(40)
Naturally, the equation \(39\) is reduced to the result in the mean-field approximation if \(\Sigma_{s2} = 0\) and \(\Sigma_{v2} = 0\). Putting Eq. \(39\) into Eqs. \(26\) and \(27\) we can obtain the equation of motion for \(\sigma\) and \(\omega\) mesons. Naturally, the thermodynamical consistency of Eq. \(39\) is trivial because of the equations of motion for mesons.
4 Summary

In summary, we have studied the density-dependent effective potential in nuclear matter using the QHD and the nonperturbative evolution equation. The method for calculating the explicit $\rho$-dependence of the effective potential is formulated. Using this method, the analytical form of the energy density in nuclear matter is derived. Although our approximation used here is a simple one, the result is beyond the ordinary mean field approximation which is usually used. Our result is consistent with the thermodynamical relation because of the equation motion for mesons. It is interesting to solve the evolution equation numerically in more realistic approximation.

References