Topics on noncanonical representations of Gaussian processes

By

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Abstract: In this article, we give a survey of recent studies for noncanonical representations of Gaussian processes, and improve some results. Especially, we find applications of noncanonical representations. We also consider noncanonical representations of a Gaussian semimartingale so as to be independent of an infinite-dimensional subspace.

Key Words: Gaussian processes; Canonical representations; Brownian motions; Müntz-Szász theorem; Semimartingale.

1 Introduction

Let a Gaussian process $X = \{X(t); t \geq 0\}$ be represented as

$$X(t) = \int_0^t F(t, u)dB(u), \quad (1)$$

where $B$ is a Brownian motion and $F(t, \cdot) \in L^2(0, t)$ is a nonrandom representation kernel. Always $\mathcal{B}_t(X) (\equiv \sigma\{X(s); s \leq t\})$ is smaller than or equal to $\mathcal{B}_t(B)$ for each $t > 0$. There are many such representations for the process $X$. Among such representations, if $\mathcal{B}_t(X) = \mathcal{B}_t(B)$, the representation (1) is said to be canonical with respect to $B$. We remark that, for a Gaussian process, $\mathcal{B}_t(X) = \mathcal{B}_t(B)$ is equivalent to $H_t(X) = H_t(B)$, where $H_t(X) (\equiv LS\{X(s); s \leq t\})$ is a closed linear span of $\{X(s); s \leq t\}$. The canonical representation is uniquely determined if it exists.

The concept of canonical representation was originated by Lévy [12]. For canonical representation, we consider that the past of $X$ and that of $B$ have the same information.
In this case, the randomness of $X$ is fully expressed by $B$. This Brownian motion $B$ is called an innovation of $X$.

Conversely, for noncanonical representations, the information of the past of $X$ is smaller than that of $B$. We take notice of the gap of them. The gap $H_t(B) \ominus H_t(X)$ may characterize the noncanonical property of the representation. Although noncanonical representations have been considered to be junk, we will find their applications in Section 2.

Concerning to the existence of canonical representation, Hida [7] introduced the concept of multiplicity. In general, the multiplicity of the process may be infinite. Examples of the multiplicity more than one are given in [8]. Recently, Hitsuda [10] has been considering a condition for a Gaussian semimartingale to have unit multiplicity. If the multiplicity equals two, the information of the process will be split into two (infinite-dimensional) subspaces. Namely, the remainder removing information of the first process is still infinite-dimensional. In Section 4, we consider the possibility of the existence of noncanonical representations of a Gaussian semimartingale so as to be independent of a given infinite-dimensional subspace.

In Section 5, we consider stationary Gaussian processes. From the viewpoint of stationary processes, we will realize that the result in Section 4 is just a special case.

### 2 Noncanonical representations

For any $N \in \mathbb{N}$, we can construct a noncanonical representation of a Brownian motion so as to be independent of a given $N$-dimensional subspace. Let $g_1, g_2, \ldots, g_N \in L^2_{loc}[0, \infty)$ be linearly independent in $[0,t]$ for any $t > 0$. For these functions $g = \{g_1, \ldots, g_N\}$, put a Volterra-type integral operator $K_g : L^2[0, \infty) \to L^2[0, \infty)$ by

$$K_g \alpha(s) = \int_0^s \sum_{i,j=1}^N g_i(s) \Gamma^{ij}(s) g_j(u) \alpha(u) du, \quad \alpha \in L^2[0, \infty),$$

where $\Gamma^{-1}(s) = (\Gamma^{ij}(s)) = \left(\int_0^t g_i(u)g_j(u) du\right)^{-1}$. The matrix $\Gamma(s)$ is invertible for any $s > 0$ since the system $g(s)$ is linearly independent. By using this operator, we can uniquely construct a Brownian motion so as to be independent of the subspace spanned by $\left\{\int_0^t g_j(u)dB(u); j = 1, 2, \ldots, N\right\}$.

**Proposition 1 ([3, Theorem 2.1])** Define $B_g$ by

$$B_g(t) = B(t) - \int_0^t \left(\int_0^s \sum_{i,j=1}^N g_i(s) \Gamma^{ij}(s) g_j(u) dB(u)\right) ds$$

(3)
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\[
\begin{align*}
&= \int_0^t \left( 1 - \int_s^t \sum_{i,j=1}^N g_i(s) \Gamma^{ij}(s) g_j(u) ds \right) dB(u) \quad (4) \\
&= \int_0^t \left( I - K^*_g \right) 1_{(0,t)}(u) dB(u), \quad (5)
\end{align*}
\]

where \( K^*_g \) is the formal adjoint operator of \( K_g \). Then \( B_g \) is a Brownian motion and is noncanonical with respect to \( B \) satisfying

\[
H_t(B) = H_t(B_g) \oplus LS \left\{ \int_0^t g_j(u) dB(u); j = 1, 2, \ldots, N \right\}. \quad (6)
\]

We prefer to write (5) symbolically by

\[
\dot{B}_g(t) = (I - K_g) \dot{B}(t), \quad B_g(0) = 0. \quad (7)
\]

In general, for a given Gaussian process, it is difficult to find the concrete form of canonical representation. Using the proposition above, we are able to obtain the canonical representation by the use of a noncanonical representation being finite codimensional, as follows:

**Proposition 2 ([6, Theorem 2])** If a Gaussian process \( X \) has a noncanonical representation (1) satisfying

\[
H_t(B) \oplus H_t(X) = LS \left\{ \int_0^t g_j(u) dB(u); j = 1, 2, \ldots, N \right\}, \quad (8)
\]

then \( X \) has unit multiplicity, and

\[
X(t) = \int_0^t \tilde{F}(t, u) dB_g(u) \quad (9)
\]

is a canonical representation of \( X \) with respect to \( B_g \), where \( B_g \) is defined by (4) and \( \tilde{F}(t, \cdot) = (I - K_g) F(t, \cdot) \).

When we get a representation (1) of \( X \), we naturally consider whether the representation is canonical or not. Namely, we check the orthogonal complement \( H_t(B) \ominus H_t(X) \). If we find (8), we can obtain the canonical representation thanks to the proposition above. However, there is a possibility that the orthogonal complement \( H_t(B) \ominus H_t(X) \) may be infinite-dimensional.

**Remark 1** Letting \( N \) in Proposition 1 tend to infinity in order to consider noncanonical representation independent of an infinite-dimensional subspace, we are confronted with the problem whether the subspace \( LS \{ g_n, n \in \mathbb{N} \} \) span the whole \( L^2 \)-space. In the case of \( g_n(t) = t^{\nu_n} \), the problem whether \( \{ t^{\nu_n}, n \in \mathbb{N} \} \) is complete or not is known as the M"untz-Sz"asz theorem [2].
As an application of Proposition 1, let us consider a bridge \( P \) over a Gaussian process \( X \), defined by \( P(t) := X(t) - E[X(t)|X(T)] \) for \( T > 0 \).

**Theorem 3** Let \( X \) have the canonical representation (1). The innovation of the bridge \( P \) is the Brownian motion \( \tilde{B} \) defined by (4) by putting \( N = 1 \), \( g_1(u) = F(T, u) \) in Proposition 1. Namely, the bridge \( P \) has a canonical representation with respect to \( \tilde{B} \)

\[
P(t) = \int_0^t G(t,u) d\tilde{B}(u),
\]

for some \( G \).

**Proof:** From the definition of \( P \),

\[
H_t(P) = H_t(X) \ominus LS\{X_T\} = H_t(B) \ominus LS\{X_T\}
\]
since \( X \) is canonical with respect to \( B \).

On the other hand, the Brownian motion \( \tilde{B} \) has the same filtration as \( P \), because

\[
H_t(\tilde{B}) = H_t(B) \ominus LS\{\int_0^t F(T,u) dB(u)\}
\]

\[
= H_t(B) \ominus LS\{X(T)\}
\]

\[
= H_t(P)
\]
since \( \int_0^T F(T,u) dB(u) \) is independent of \( H_t(B) \).

Therefore, \( \tilde{B} \) is the innovation of \( P \). \( \square \)

**Remark 2** The problem to determine the concrete form of \( G \) remains open.

### 3 Iteration of the operators \( K_g \)

Here we show another idea to construct noncanonical representations of a Brownian motion so as to be independent of a finite-dimensional subspace according to [4].

For any \( g \in L^2_{\text{loc}}[0, \infty) \), define the operator \( K_g : L^2[0, \infty) \rightarrow L^2[0, \infty) \) by

\[
K_g \alpha(t) = \int_0^t \frac{g(t)}{\int_0^t g(u)^2 du} g(s) \alpha(s) ds, \quad \alpha \in L^2[0, \infty).
\]

This is the case of \( N = 1 \) in (2).
Define $B_n$ by $\dot{B}_n(t) = (I - K_g)^n \dot{B}(t)$, for $n \in \mathbb{N}$. Then $B_n$ is again a Brownian motion and is noncanonical with respect to $B$ satisfying

$$H_t(B) = H_t(B_n) \oplus \bigoplus_{k=1}^{n} LS \left\{ \int_0^t (I - K_g^*)^{k-1} (1_{(0,t)}g)(u)dB(u) \right\}.$$  

Thus we can construct a noncanonical representation of a Brownian motion $B_n$ so as to be independent of an $n$-dimensional subspace.

However, even letting $n$ tend to infinity, we cannot obtain a noncanonical representation independent of an infinite-dimensional subspace. In order to construct a noncanonical representation independent of an infinite-dimensional subspace, we will need another idea.

**Proposition 4 ([4, Theorem 2.3])** For any $g \in L^2_{\text{loc}}[0, \infty)$,

$$H_t(B) = \bigoplus_{k=1}^{\infty} LS \left\{ \int_0^t (I - K_g^*)^{k-1} (1_{(0,t)}g)(u)dB(u) \right\}.$$  

Incidentally, we quote some results for the operator $K_g$, for an information.

**Proposition 5 ([4, Theorem 2.2])** For $g, h \in L^2_{\text{loc}}[0, \infty)$, $B_{g,h}$ defined by

$$\dot{B}_{g,h}(t) = (I - K_h)(I - K_g) \dot{B}(t), \quad B_{g,h}(0) = 0,$$

is a Brownian motion and is noncanonical with respect to $B$ satisfying

$$H_t(B) = H_t(B_{g,h}) \oplus LS \left\{ \int_0^t (I - K_g^*)(1_{(0,t)}h)(u)dB(u) \right\} \oplus LS \left\{ \int_0^t g(u)dB(u) \right\}.$$  

We note that $B_{g,h}$ is not symmetric with respect to the functions $g$ and $h$. Next, we give a condition for the operators $K_g$ and $K_h$ to be commutative, which is naturally symmetric with respect to $g$ and $h$.

**Theorem 6 (cf.[4, Theorem 3.1])** For $g, h \in L^2_{\text{loc}}[0, \infty)$, the operators $K_g$ and $K_h$ are commutative if and only if there exist constants $\beta > 0$ and $C > 0$ such that

$$\int_0^t h(u)^2 du = C \left( \int_0^t g(u)^2 du \right)^{\beta}, \text{ for any } t > 0.$$  

**Corollary 7** If $g(t) = t^p$ and $h(t) = t^q$, $p, q > -1/2$, then $K_g$ and $K_h$ are commutative.
4 Semimartingale and Infinite-dimensional orthogonal complement

We consider a Gaussian semimartingale given by

\[ X(t) = cB(t) - \int_0^t \int_0^s k(s, u)dB(u)ds, \]

where \( c \neq 0, k(s, \cdot) \in L^2(0, s) \) for any \( s > 0 \), and

\[ \int_0^t \left( \int_0^s k(s, u)^2du \right) \frac{1}{2} ds < \infty, \text{ for any } t > 0. \]

If \( k(\cdot, \cdot) \in L^2((0, t)^2) \) for any \( t > 0 \), then \( X \) is equivalent to \( B \), i.e. the distributions of \( X \) and of \( B \) are mutually absolutely continuous [9]. In this case, the representation (10) is a fortiori canonical with respect to \( B \). Though (3) is of the form (10), it is noncanonical and the kernel function does not belong to \( L^2((0, t)^2) \).

Suppose \( X \) is represented as

\[ X(t) = \int_0^t f(u/t)dB(u), \text{ where } f \in L^2(0, 1). \]  

(11)

Concerning to the condition that (11) is of the form (10), the following fact is known.  

**Proposition 8** ([11, Theorem 6.5]) The condition that (11) is semimartingale in the filtration of \( B \) is

(i) \( f(1) = c \neq 0 \),

(ii) \( \int_0^1 v^2 f'(v)^2du < \infty \).

From (i) and (ii), we can put

\[ f(x) = c - \int_x^1 \frac{1}{v} \varphi(v)dv, \text{ where } \varphi \in L^2(0, 1). \]

Suppose a Gaussian semimartingale \( X \) is given by

\[ X(t) = \int_0^t \left\{ c - \int_{u/t}^1 \frac{1}{v} \varphi(v)dv \right\}dB(u), \]

(12)

where \( \varphi \in L^2(0, 1) \) and \( c \neq 0 \).

Here we shall consider whether there exists a noncanonical representation of \( X \) having an infinite-dimensional orthogonal complement \( H_t(B) \ominus H_t(X) \) spanned by \( \{ \int_0^1 u^{q_n}dB(u); n \in \mathbb{N} \} \).

As we have stated in Remark 1, it is enough to consider the case where \( \{u^{q_n}; n \in \mathbb{N} \} \) does not span the whole \( L^2 \)-space.
Proposition 9 (Müntz) For a sequence \( \{q_n\} \) with \( 0 = q_0 < q_1 < q_2 < \ldots \), the system \( \{u^{q_n}; n = 0, 1, 2, \ldots\} \) is not complete if and only if
\[
\sum_{n=1}^{\infty} \frac{1}{q_n} < \infty.
\]

Theorem 10 Let a Gaussian process \( X \) be of the form (12). Then, for a sequence \( \{q_n\} \) with \( 0 = q_0 < q_1 < q_2 < \ldots \), \( X \) never satisfies
\[
H_t(X) \perp \left\{ \int_0^t u^{q_0} dB(u); n \in \mathbb{N} \right\} \text{ in } H_t(B).
\]

Proof: Suppose (13) holds.

If the relation \( H_t(X) \perp \int_0^t u^{q} dB(u) \) is satisfied,
\[
\int_0^t \left\{ c - \int_{u/t}^1 \frac{1}{v} \varphi(v) dv \right\} u^q du = 0, \text{ for any } t > 0.
\]
This is reduced to
\[
c = \int_0^1 x^q \varphi(x) dx.
\]
By the Schwarz inequality, we have
\[
|c| \leq \frac{1}{\sqrt{2q + 1}} \|\varphi\|_{L^2(0,1)}.
\]
Therefore \( q_n \) never tends to infinity if (13) holds. However, \( q_n \) tends to infinity because of Proposition 9. This is a contradiction.

5 Stationary Gaussian processes

It is well-known that, for \( X \) represented as (1), if the representation kernel \( F \) is homogeneous, then \( X \) can be transformed to a stationary process. Since the representation kernel of (12) is a homogeneous function of degree 0, (12) is transformed to a stationary process
\[
Y(s) := \frac{1}{\sqrt{2e^s}} X(e^{2s}), \quad s \in \mathbb{R},
\]
\[
= \int_{-\infty}^s e^{-(s-u)} \left\{ 1 - 2 \int_0^{s-u} \varphi(e^{-2v}) dv \right\} dW(u), \quad (14)
\]
where \( dW(u) = \frac{1}{\sqrt{2e^u}} dB(e^{2u}) \) is a Wiener measure.
Remark 3  This transformation preserves canonical property; that is to say,

\[ H_t(X) \perp \int_0^t u^q dB(u), \quad q > -1/2, \]

is corresponds to

\[ H_s(Y) \perp \int_{-\infty}^{s} e^{(2q+1)u} dW(u). \]

It is known that if the Fourier transform of the representation kernel of \( Y \) has a zero-point in \( \mathbb{C}_+ = \{ \Re z > 0 \} \), then the representation (14) is not canonical. More precisely, if the Fourier transform of the representation kernel of \( Y \) has a zero-point \( z_0 \in \mathbb{C}_+ \),

\[ H_s(Y) \perp \int_{-\infty}^{s} e^{-iz_0 u} dW(u) \text{ in } H_s(W). \]

We can say that number of zero-points in \( \mathbb{C}_+ \) corresponds to the dimension of the orthogonal complement \( H_t(W) \ominus H_t(Y) \).

The correspondence of Proposition 9 for thus complex case is the following:

**Proposition 11 (Szász)** For a complex sequence \( \{q_n\} \) with \( \Re q_n > -1/2 \), the system \( \{u^{q_n}; n \in \mathbb{N}\} \) is not complete in \( L^2 \) if and only if

\[ \sum_{n=1}^{\infty} \frac{\Re(2q_n + 1)}{1 + |2q_n + 1|^2} < \infty. \]

By noting what we mentioned in Remark 3, Theorem 10 is corresponding to only the case where the zero-points are on the pure imaginary-axis and are monotonically divergent. What we can conclude from Theorem 10 is thus:

**Corollary 12** If the representation (14) of a stationary process \( Y \) is noncanonical with respect to \( W \) with an infinite-dimensional orthogonal complement \( H_s(W) \ominus H_s(Y) \), then the imaginary part of zero-points of the Fourier transform of the representation kernel of (14) cannot tend to infinity.

As we have pointed out in [5], the problem even in the case where the zero-points on the pure imaginary-axis tend to zero is still open, to say nothing of the case where infinitely many zero-points are lying horizontally. It seems that the authors of [1] and [13] ignore these cases.
References


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